MECHANICAL INTEGRATORS,

INCLUDING

THE VARIOUS FORMS

OF

PLANIMETERS.

BY

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PREFACE.

Mechanical aids to mathematical computation have always deservedly been regarded with interest.

Aside from the labor-saving quality which most of them possess, they have a value arising from the fact that they represent thoughts of more or less complexity expressed in mechanism.

They are of many kinds, and serve widely different purposes. The reader will find in this essay descriptions of many that are useful directly or indirectly to engineers.

Mechanical Integrators.

All measurements are made in terms of some fixed unit. The method may consist of a simple comparison of the unit with the quantity to be measured, but when this cannot conveniently be done, some indirect means must be employed. Indirect measurements may be made by measuring some physical effect, the magnitude of which is known to be a function of the quantity to be measured; as, for instance, when the length of a rod or wire is estimated by its weight. Where, however, the unit in terms of which the measurement has to be made, is what is known as a derived unit, the indirect method generally consists in measuring in terms of the simple units from which the

former is derived, and performing, with the results, the necessary calculation. An example of this latter method is given in obtaining the contents of an area, by taking its length and breadth and multiplying them together, instead of adopting the tedious process of ascertaining, by direct comparison, how many times the unit of area would be contained within it.

Now, such calculations, even when of so simple a kind as mere multiplication, often become very inconvenient, and a large number of instruments have been designed for performing them by mechanical means. Such instruments may be divided into two classes, one in which the final result of conditions which vary in an arbitrary manner is found, such as the contents of a surface or the work of a motor, requiring a process of multiplication or addition; the other in which the relation or ratio, at any instant, of two such quantities is given, such as space and time in the case of velocity, requiring at each instant a process of

division. The object of the present paper is to deal with the theory, design and practical applications to engineering problems of the former class alone. It may be briefly stated that very little has yet been practically done in the use of the latter. Quite recently, Professor A. W. Harlacher, of Prague, has published an account of the instruments and methods of Harlacher, Henneberg and Smreker, for gauging the velocity of a river current, the principle of which is the same as that independently adopted by the author and others in this country.

The conditions or data above referred to, from which the required result has to be calculated by instrumental means, are obtained in two ways:

(1) By intermittent or separate observations and measurements.

(2) By the continuous motion of a machine in connection with self-recording apparatus.

The former is the case in measuring an area of country, taking dimensions of a river or embankment section, or ob-

taining the forces exerted at different times by a machine or body in motion. The latter is generally given in the form of a graphic record, an important example of which is the diagram of energy or work taken from a prime mover. In both cases the result, whatever it be, whether boundary, area, volume, work, etc., can be found by calculation, but only with an approximation to the truth, depending upon the extent of the calculation. The reason of this is that the data of calculation, which are taken directly in the first case, or selected from the graphic record in the second, only represent actual conditions more or less closely, according as the number of data so taken is greater or less, and the greater the number the greater is the labor of determining the result. The instruments discussed in this paper perform such work mechanically, with the great advantages of rapidity of operation, accuracy of results, and without requiring mental effort on the part of the manipulator; and all this, moreover, to a great

extent independently of the complexity of the calculation required. All the results the measurement of which will be considered, can be measured graphically. If the observations have been made separately, they can be plotted, whether in the form of a diagram of energy, or on the plan or elevation of an area or section, and the boundary can be filled in with a tolerably close approximation to accuracy. In the other case the graphic record is, or may be, directly given. The subject, as far as the theory of the calculation goes, can therefore be studied with reference to such diagrams without the necessity of considering in the first case how they were obtained, and it will be convenient to do this, and afterwards to examine separately various examples of their application. Such diagrams may be drawn upon any kind of surface, and an instrument for dealing with measurements upon that of a sphere will be hereafter described. A plane surface may, however, be employed upon which to represent all cases of any practical import-

ance, and the question thus arises, What are the measurements of the nature under consideration which are required, and which can be obtained from either a regular or an irregular plane of figure?

Such measurements are of three kinds:

(1) The length of its perimeter or boundary.

(2) The area of its superficial contents.

(3) Its relation to some point, line, or other figure on the surface, *e. g.*, its moment of area or moment of inertia about a given line.

All these three kinds of quantities can be ascertained by successive operations of addition. The first requires the addition of elements of length, the second may be obtained by adding up successive elements in the form of strips of area, and the third by adding products obtained by multiplying such strips by some quantity, the magnitude of which depends upon the position of the other point, line, or figure in question. Taking the general case of an irregular fig-

ure, it is evident that absolute accuracy can only be obtained when this operation becomes that of integration or summing up of an infinite series of indefinitely small quantities. Instruments for performing this operation are therefore called "mechanical integrators." In all such instruments the rolling action of two surfaces in frictional contact is employed, for this, as will be hereafter seen, enables the conditions of motion to be continuously varied in a way which could not be effected by mere trains of wheel-work, such as form the mechanism of some kinds of calculating machines. This fact necessitates something more than a mere discussion of the mathematical principles upon which the calculations are performed, for though the action of an integrator may be absolutely correct as far as its theory of the performance of the calculation is concerned, yet there is always some instrumental error depending upon the rolling, and also, as will be seen, of the slipping of the two surfaces in frictional contact.

This error may be exceedingly small, but it is a matter of great importance to ascertain its exact amount, and the subject will therefore be investigated at length, under the heading "Limits of Accuracy of Integrators," where an account will be also given of the experimental results of Professor Lorber, of Leoben; Dr. William Tinter, of Vienna, and Dr. A. Amsler, of Schaffhausen. In this investigation it will be shown that when integrators are examined upon the mechanical principles of action, they are all found to belong to one of two classes.

(1) In which the surfaces in question slip over each other.

(2) In which only pure rolling motion of the surfaces is assumed to take place.

The significance of this mode of classification is that it not only leads to a clear understanding of the nature of the results to be expected from any particular instrument, and teaches the best method of manipulating it, with regard to its position relatively to the figure to be measured, but it also brings out prominently

the mechanical principle upon which the inventor has relied sometimes, as it would appear, unconsciously, for the accuracy of the results expected to be obtained.

It may be here remarked that the same principle, by which an integrator is employed to determine a result from an autographic record, may be applied directly to obtain a continuous result from the machine or body in motion, such, for instance, as an ordinary dynamometer or dynamo-electric machine, from which the autographic record was obtained. Thus, after discussing the action of integrators for dealing with diagrams, it will only be necessary to consider the mechanical details of the instruments for direct application to a machine, and this will be done under the head of "Continuous Integrators."

Coming now to the consideration of the actual measurement of the three kinds of quantities, it will be found that the first is very simple, and, in fact, the only reason why it is not convenient to meas-

ure it, by comparing it with the unit of length in the ordinary way, is its continuous change of direction.

The only mechanical method of rectifying a curve, as it is called, that is, obtaining its length as a right line, is by rolling a wheel along it. This wheel is connected with a suitable train of wheels for recording the total number of revolutions, and as the rolling circle of the first wheel is either a unit in length or contains a known relation to this unit, the length of the curve or boundary is given at once by the reading of the graduated wheels. The use of such instruments is very ancient, and Beckmann, in his "History of Inventions," describes amongst various odometers, one mentioned in the Tenth Book of Vitruvius. Such an instrument is made upon a small scale for use by a draftsman, and in one form it is sometimes termed an "opisometer," in another form the chartometer, or Wealemefna; it is also employed upon a large scale as a road or route measurer. The same principle has been em-

ployed in one of the latest forms of anemometer, in which the plane of the wheel is always kept coincident with the direction of the wind, while its edge rolls in contact with the recording surface, and measures the total travel of the wind. In this class falls the device suggested in a letter to *Nature* by Mr. V. Ventosa, of Madrid, for continuously obtaining the N., E., S. and W. components of the wind, a device which was independently arrived at by the author in connection with certain mathematical principles referred to hereafter.

The object of employing a rolling wheel is merely to enable the direction to be changed so as always to coincide with the curve to be measured, and the principle is, therefore, that of direct unit measurement or comparison. The foregoing instrument, which evidently belongs to the second class, in which only rolling motion, without slipping, is supposed to take place, forms, however, one kind of mechanical integrator.

The measurement of the other two

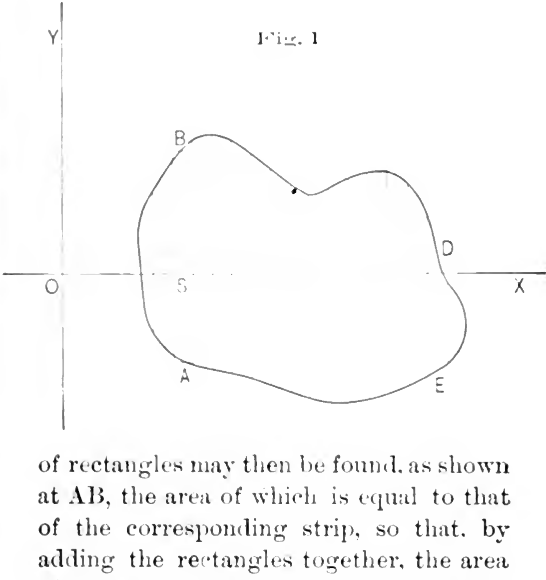
kinds of quantities is, as in the case of the first, a problem of addition; but, instead of being the addition of an infinite number of infinitely short lines, an infinite number of continually changing magnitudes has to be added. This is the same both for instruments required in the second kind of measurement, or "Area Planimeters," and those required in the third kind, or "Moment Planimeters." From the fact that the changing magnitudes referred to are simpler to deal with in the case of calculating the contents of areas than in finding their moments or any mathematical results of an equivalent kind, the discussion of the theory of the two above kinds of instruments is not the same, and will, therefore, be dealt with under two separate headings.

Area Planimeters.

The area of any plane figure, such as ABDE (Fig. 1), can be obtained in the following manner. Take any straight lines, OX and OY, at right angles to each

other, and parallel to OY; draw a series of straight lines equidistant from each other, dividing the figure into a number of strips, or elements of area. A series

Fig. 1



of rectangles may then be found, as shown at AB, the area of which is equal to that of the corresponding strip, so that, by adding the rectangles together, the area of the whole figure is obtained, as adopted in the common method of finding the area of an indicator diagram. The greater

the number of strips the more closely will the height of the two sides of each tend to become equal to each other, and to the height of the corresponding rectangle.

Let Δx = the width of any element of area such as AB, at a distance x from O.
y = height of the corresponding rectangle.

Then $y \Delta x$ = area of the element AB, and the sum of all such elements of area is the area of the figure ABDE. When the number of elements is increased indefinitely, this expression becomes

$$\int_a^b y dx = \text{area of the figure ABDE},$$

a and b being the extreme values of x, *i.e.*, the limits of integration.

It is evident, therefore, that the area will be correctly measured by an instrument in which the recording wheel or measuring roller always turns at a rate proportional to the ordinate y of the curve, while the body from which it derives its motion moves at a uniform rate along the axis OX.

Area planimeters have been classified according to apparently different modes by which the operation of integration is performed; but, since the action of them all can be explained upon the foregoing principle of adding elements of area, and, in fact, by means of the same notation, it is not surprising that such classifications are anything but satisfactory. In fact, in one important kind of planimeters, it becomes doubtful to which class they belong, or, whether they should not be placed in two or more classes. It is, without doubt, very convenient to distinguish different planimeters, and, therefore, the names which have been given them will be used; but this will not denote any difference of principle, and the classification which will be adopted is that already explained, and depends on mechanical conditions of action. In what follows, one mode of viewing the mathematical operation is adhered to throughout, and it may be stated that the object of the author has been to make clear the principles of action of integrators, rather

than to obtain rigid and exhaustive demonstrations of their theory.

Planimeters in which Slipping of the Measuring Roller Takes Place.

From a brief account of the subject by Professor Lorber, it appears that the first recorded idea of a planimeter is attributed to Hermann, of Munich, who worked it out with Lämmle. This idea of Hermann's, which was published in 1814, seems to have fallen into oblivion, for in 1827 Oppikofer, of Berne, constructed a planimeter upon similar principles, and it was thenceforth called after his name. On the other hand, Favora gives the priority to Professor T. Gonella, of Florence, who, in 1828, without any knowledge of what Hermann had done, invented and described a very similar instrument. The development of the planimeter seems to have grown out of the instrument of Oppikofer, who, in conjunction with a Swiss mechanic, Ernst, finished a planimeter which won a prize, in Paris, in 1836. Important improvements are due

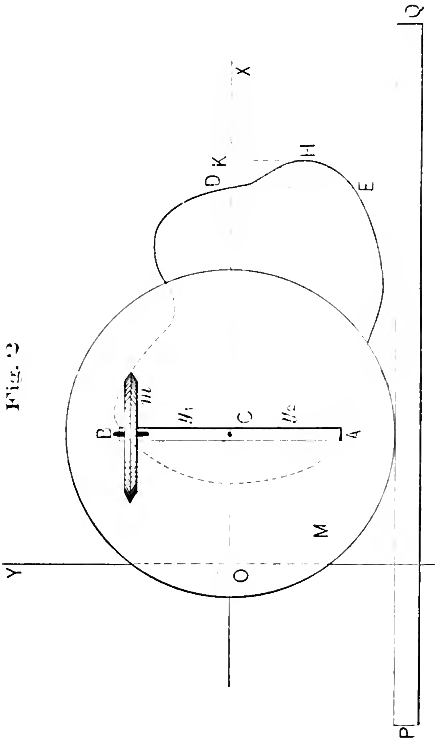


Fig. 2

to Wettli, of Zurich, who, with Starke, in 1849, took out a patent in Austria for the instruments now called the Wettli-Starke planimeter. Later on, in England, other inventors (Sang, Moseley) worked at the subject, but all these instruments depended for their action on the same principle, which is as follows:

Let M (Fig. 2) be the plan of a disk rolling in contact with a straight guide PQ, which is parallel to OX, and at a distance from it equal to the radius of the disk, so that the plan of the center of the latter always lies in OX. Let m be a roller upon the surface of the disk, graduated and connected with wheel-work and an index, so that the distance turned through over the surface of the disk can be read in revolutions or parts of a revolution. The plan of the point of contact (B) of the roller with the disk is always made to coincide with that particular point on the curve which is in the line drawn at right angles to OX, through the center C of the disk. The plane of otation of (m) which may be called the

measuring roller, is always perpendicular to the disk M, and the plan of its axis, as shown in the figure, is always parallel to OY, so that, in following the curve, it slips backwards or forwards across the surface of the disk, in a direction parallel to OY. Suppose the disk to roll along PQ for a distance Δx, equal to the width of the element AB.

Then if y_1 = distance of B from OX.
R = radius of disk M.
r = radius of measuring roller m.
n_1 = consequent reading of measuring roller for this travel of disk.

Then $\frac{y_1}{R}\Delta x$ = linear distance turned through by a point on the disk at the distance y_1 from the center.
$2\pi r n_1$ = linear distance turned through by a point on the circumference of m; but since m rolls on M these distances are equal.

Therefore $$2\pi r n_1 = \frac{y_1}{R}\Delta x,$$

or, $$n_1 = y_1 \Delta x \times \frac{1}{2\pi r R};$$

but $\frac{1}{2\pi r\mathrm{R}}$ is a constant, which, by taking r and R in suitable ratio may be made unity.

Then $$n_1 = y_1 \Delta x,$$
that is, the reading of the roller m measures that part of the area of the element above OX.

If the point of contact be made to follow round the curve continuously in one direction, then, when the portion of AB below OX is being measured, the disk is moving in the opposite direction along PQ, but, at the same time, the roller is turning in the opposite way relatively to the disk to that which it was doing before, since the point of contact is now below C. The final result of these two opposite motions is to cause the roller to turn, as at first, and so add the result given for CA to what was given for CB. If the motion of the disk Δx for the width of AB be now regarded as negative, and $-y_2$ = distance AC

also n_2 = reading of roller for this element of area,

then by similar reasoning to that already used,

$$n_2 = (-y_2)(-\triangle x)$$
$$= y_2 \triangle x,$$

and $n = n_1 + n_2 = (y_1 + y_2) \triangle x = y \triangle x =$
$=$ area of element AB.

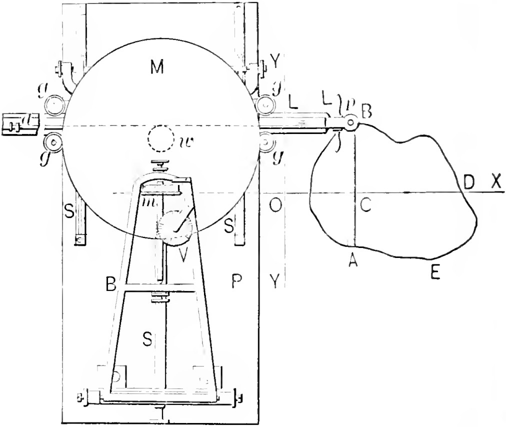
This reasoning holds for any possible position of the roller, or of the axis OX, which may be altogether outside the figure, as it practically is for the integration of the portion DHE. Then it will be found that DKH is subtracted, and DKHE is added, so as to give the required actual area DHE.

Inasmuch as this reasoning is independent of the actual value of the width of the element, and as the vertical motion of the roller m has no effect in theory upon the distance rolled through by it, therefore in the limit when $\triangle x$ becomes infinitely small, the actual value of the series of infinitely narrow strips which compose the figure ABDE is given by the final reading of the roller when the traverse of the boundary is completed.

The Wettli-Starke planimeter (Fig. 3)

acts directly upon this principle, with the exception that it is the disk that is moved according to the changes in y instead of the measuring roller, and the

Fig. 3



following is a description of the best form of this instrument:

On a base-plate P (Fig. 3), are three guides SSS, along which a frame carrying the vertical axis of the disk M can be moved to and fro. The disk, which is

made of glass and covered with paper, has two motions, one rectilineal along the guides, and one of rotation about its axis. The motions are imparted to it by means of an arm (L), which passes through the roller-guides (*gg*) in the frame carrying the disk, and rotates the latter by means of a German silver wire (*dd*) passing round a cylinder *w* upon its axis, and attached by the two ends to the extremities of the arm. The measuring roller (*m*) rests upon the surface of the disk, being carried in another frame (B), which is hinged to the base-plate. The action is as follows: the base-plate being placed in juxtaposition to the figure to be integrated, any line parallel to the guides, *i.e.*, to the direction of rectilinear motion of the disk, may be taken as the axis OY; and line OX, drawn through the edge of the roller, perpendicular to OY, may be taken as the other axis. Then, as the pointer *p* at the extremity of the arm is made to pass round the boundary of the figure, the disk will be turned through a distance proportional

to the travel along OX, while at any instant the roller (m) is at the same distance from the center of the disk as the pointer is from OX.

If y_1 = CB = mean height of element Δx = = width of element AB.

Then, by the reasoning already given, the reading of the roller which the pointer passes over the upper boundary of the element AB, is

$$n_1 = y_1 \Delta x,$$

and the final reading of the roller is

N = area of the figure ABDE.

Hansen, in 1850, still further improved this instrument, and, in conjunction with Ausfeld, introduced a different method of reading the result, and of carrying the frame, this instrument being known as the Hansen-Ausfeld planimeter. Various other instruments of the same kind were shown in the Great Exhibition of 1851, but in all, the motion of the arm carrying the pointer was "linear;" that is, the motion, which must be possible in every direction, is obtained by compounding two rectilinear movements, at right angles

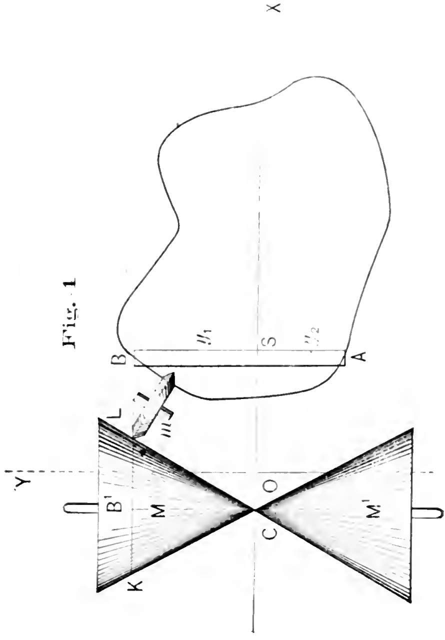


Fig. 1

to each other. Such instruments are therefore called "linear planimeters."

Many different forms of linear planimeters have been suggested, but the only modification of the disk and roller which it will be worth while to notice is the cone and roller.

Let MM′, Fig. 4, be the cone corresponding to the disk M, and rolling on the edge of its two bases in a direction parallel to OX. Let the roller m always be in contact with a circle on the cone, whose center B′ is at a distance CB′ from the apex C of the cone, such that

CB′ = SB = y = mean ordinate of element SB.

where the element AB is being at that instant integrated. Adopting the same notation as before, when the cone has rolled over the surface through a distance $\triangle x$, then, whatever be the angle of its apex, the distance rolled through by the roller m is

$$2\pi r n_1 = \frac{y_1}{R} \triangle x,$$

or

$$n_1 = y_1 \triangle x \times \frac{1}{2\pi r R}.$$

As might have been anticipated, the expression is the same as was obtained in the case of the disk, the latter being a special case of the cone when the vertical angle is 180°.

Thus the cone may be employed instead of the disk, and such an instrument was invented by Mr. E. Sang, who, in 1852, published a description of it, according to which the action was extremely accurate, but it does not appear to have come into very extensive use.

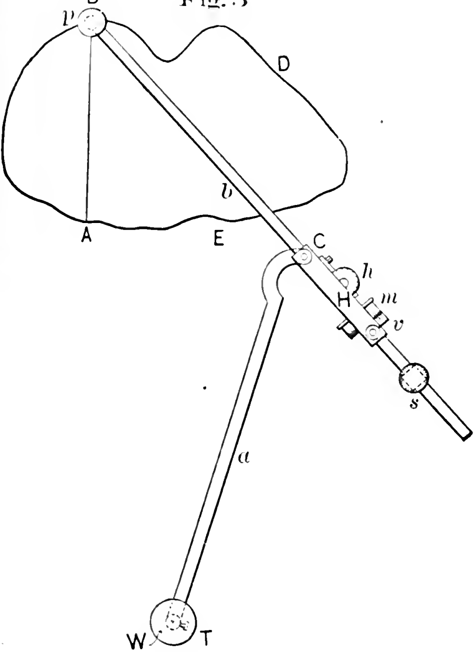
No more instruments of the kind will be described, since they have given place to those in which the arm carrying the pointer turns about a center or pole, and which are, therefore, called "polar planimeters."

In the year 1856, Professor Amsler-Laffon invented and brought before the world the now well-known polar planimeter bearing his name, and, since then, no less than twelve thousand four hundred of these instruments have been made and sent out from his works at Schaffhausen. According to authorities,

which Professor Lorber quotes, Professor Miller, of Leoben, invented independently a planimeter of this kind in the same year (1856), which, being made by Starke, of Vienna, is known as the Miller-Starke planimeter. Previous to this, in 1854, Decher, of Augsberg, as well as Bouniakovsky, of St. Petersburg (1855), had improved upon previously-existing forms of polar planimeter, though it is well to note that the planimeters already mentioned as sent to the Great Exhibition of 1851 from various parts of Europe, as Italy, Switzerland, France, and Prussia, were all linear, and no mention is made of polar planimeters in the jurors' report.

The Amsler planimeter is shown in Fig. 5. It consists of two bars, (*a*) the radius bar, and (*b*) the pole-arm, jointed at the point C. The tracing point *p*, which now coincides with the point B of the figure ABDE, is carried round the curve, and the roller *m*, which partly rolls and partly slips, gives the area of the figure; and by means of the graduated

Fig. 5



dial h, and the vernier v, in connection with the roller m, the result is given correctly in four figures. The sleeve H can be placed in different positions along the pole-arm b, and fixed by a screw s, so as to give readings in different required units. A weight at W is placed upon the bar to keep the needle-point in its place, but in instruments by some other makers T is a pivot in a much larger weight, which rests on the paper. A recent minor improvement has been to fix a locking spring to the frame, so that the roller can be held when the planimeter is raised for the purpose of reading it.

The theory of the polar planimeter may be simply deduced from that of the disk and roller thus:

Let Fig. 6 represent the same disposition of the disk M with regard to the figure ABDE, as in Fig. 2. but now let the roller m move round the edge of the disk instead of across it, its distance from OX being always the same as before, viz.:

$$Oq = SB = y_1.$$

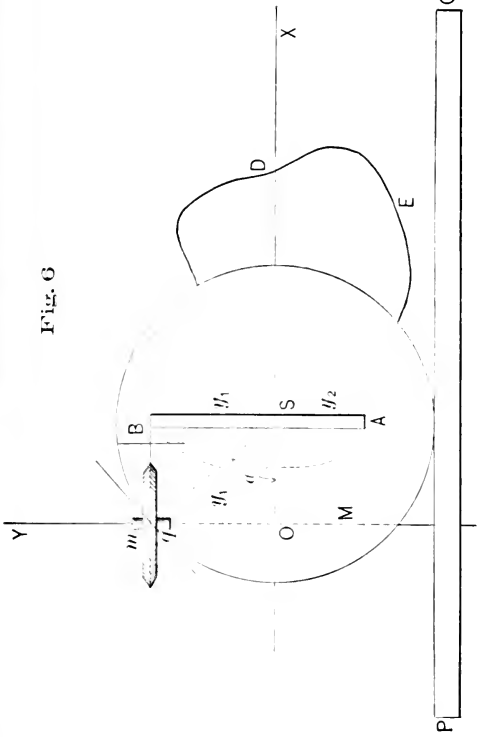


Fig. 6

The turning of m for a given travel, Δx, of the disk is found thus—draw lq (Fig 6A) tangent to the disk at m, so that

$$lq = \Delta x,$$

and draw lk parallel to the axis of rotation of m, then qk is the distance turned through, and lk is that slipped through by the edge of the roller m, when the disk has rolled through a distance Δx; therefore, using the same notation as before,

$$qk = 2\pi r n_1$$

and
$$\frac{qk}{lq} = \frac{2\pi r n_1}{\Delta x} = \sin \angle qlk;$$

but in Fig. 6 $\dfrac{Oq}{Sq} = \dfrac{y_1}{R} = \sin OSq.$

But by similar triangles

$$\angle qlk = \angle OSq = a;$$

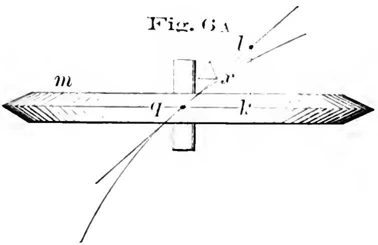
Therefore
$$\frac{2\pi r n_1}{\Delta x} = \frac{y_1}{R},$$

or
$$n_1 = y_1 \Delta x \times \frac{1}{2\pi r R}.$$

This is the same result as previously obtained, and it has been given in this way because there is an important class of planimeters to be hereafter described,

combining the polar planimeter with the disk and roller, in which a principle is employed which is thus made obvious. This principle is that the turning of m is exactly the same as if it were in contact at the point B, no matter in what posi-

Fig. 6 A

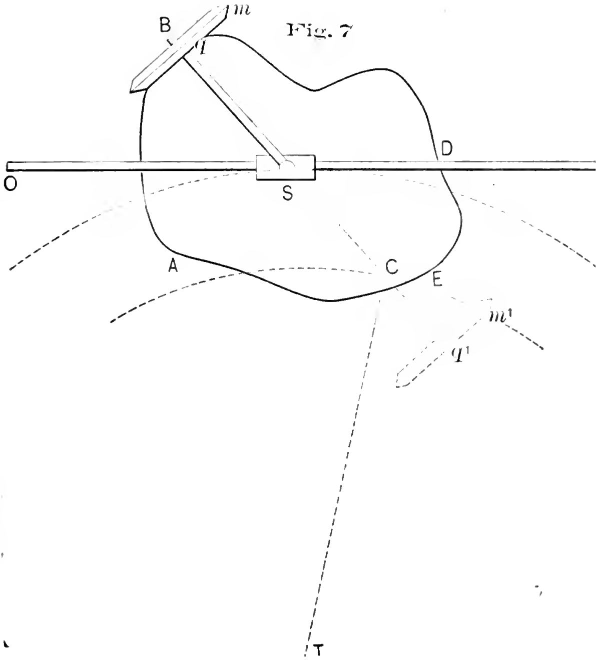


tion it may be along the line through B parallel to OX. The turning of m thus measures the area of the element as long as y does not change. If, however, the value of y changes so that m changes its distance from OX, the measuring roller is likewise turned a certain additional amount from this cause; but this does not affect the correct reading of the area so long as its first and last positions are equally distant from OX. The rea-

son is, that then the roller has turned as much in one direction in moving away from OX as it has in moving towards it, and this is the case for the initial and final positions of the pointer when the complete travel of the closed curve has been made. Now, inasmuch as the velocity of the edge of the disk is just the velocity at which the center has been shown to move along OX, the disk may be removed altogether. The roller is then moved in contact with the surface of the figure and with identically the same amount of turning as before, provided that its plane of rotation is turned so as to make the same angle with OX (which is now its direction of translation), as it did before with the direction of motion of the edge of the disk in contact with it. This is the case when it is turned through that angle, and then its axis of revolution coincides with the radius Sq.

In order to keep the direction of the plane of rotation always at right angles, it is only necessary to have a rod or bar

Fig. 7



Sq capable of turning on a pin at the point S. The pin at S is attached to a sleeve, which can freely move along a guide-bar, whose direction coincides with the axis OX. By employing the bar qSq^1 itself as the axis of rotation on which m turns, the simple planimeter shown in Fig. 7 is obtained, in which the point of contact of the measuring roller is made to pass around the diagram. The turning of the roller m correctly gives the superficial contents. The roller can be moved to any position on the rod, such as shown in dotted lines, Fig. 7, without in any way affecting its resultant turning, and the former point of contact of the roller is replaced by a pointer, which is made to follow the curve instead of the roller. Professor Burkett Webb has described to the author a planimeter of this form used in the United States, known as "Coffin's" planimeter.

If, finally, the point S, instead of moving along the straight line OX, which may be considered as a portion of a circle with its center infinitely distant,

moves along the arc SZ (Fig 7), or any other arc, as, for instance, that with radius TC, the instrument becomes the ordinary Amsler planimeter (shown in dotted lines). This explanation, so far, is based upon that given by Sir Frederick Bramwell, who has further shown that the change from the motion in a straight path to that in the arc of a circle has no effect upon the ultimate reading when the complete travel around the closed curve has been made, and the arm SB has returned to initial position. The following demonstration of the truth of this appears, however, to have an advantage, in that it follows throughout the operation of integration, especially as recent planimeters are more complicated than that of Amsler.

1st. Let point S (Fig. 8), at the extremity of the radius rod, move along the broken line STUVWXYZ, and from these points draw arcs with a common radius=R, cutting the curve in points $t_1 t_2$, $u_1 u_2$, $v_1 v_2$, &c., s and z being tangent points at the end of the curve from the

points S and Z. The proof has already been given that the integration of the complete portion st_1t_2, taken separately, is given by the reading of the measuring roller; so also are given the areas of the various other portions, $t_1u_1u_2t_2$, &c. If the separate portions were integrated consecutively, any arc, such as t_1t_2, would be traversed in both directions by the measuring roller, because it would move one way around in traversing one figure and the apposite way in going around the adjacent one, and the reading due to the arc would be eliminated.

Thus the whole curve may be integrated correctly at once without going round each separate portion formed as above, even if the point S at the end of AB moves upon a broken line instead of along OX. Next, substitute a continuous curve TUVXYZ (Fig. 8A) for the broken line. This curve may be supposed to consist of an infinite number of straight portions. The infinitely small portions contained between an arc, as u_1u_2, and another very close to it, drawn

from the beginning and end of these straight portions, may, just as in the case

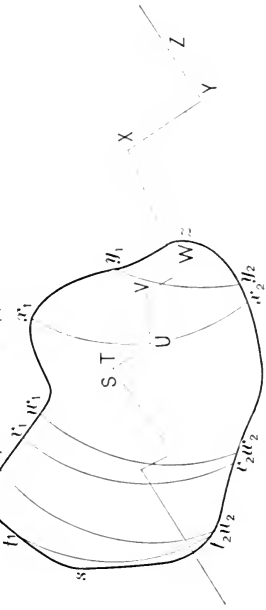


Fig. 7

of the broken line, be supposed to be integrated separately and with a correct result, which is independent even of a possible crossing of the arcs, as v_1v_2 and

$u_1 u_2$. In the same way as before, it may be seen that the arcs $u_1 u_2$, etc., need not be traversed and, so long as the point S

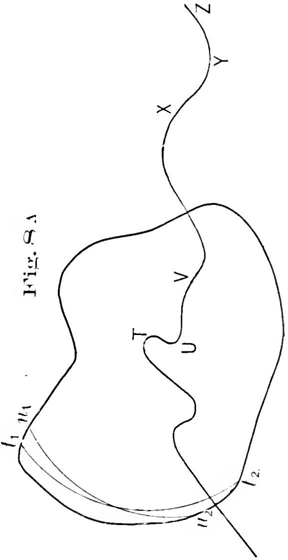


Fig. 8 A

returns to its initial position, the area of the figure is given by the simple traverse of its boundary, whatever be the curve on which the point S moves, which, in

the case of the Amsler planimeter, is a circular arc.

Various writers have explained the action of the simple polar planimeter in ways more or less different. One of these ways, recently given by Mr. F. Brooks, of Lowell, U. S., may be alluded to. He shows that the area may be treated as the difference between the area swept out by the line Tp (Fig. 9) and the sector of the zero circle, or circle upon whose circumference the pointer (p) being moved, the measuring roller is not in consequence turned. This is true, both for the outside or inside, if proper signs be taken. Let the element of area pq be passed over, the curve at p being for a small distance considered concentric with the zero circle, this small area subtending an angle w at the center; then, if values be taken as shown upon Fig. 9, in which

$CT = \text{radius-bar} = a$;

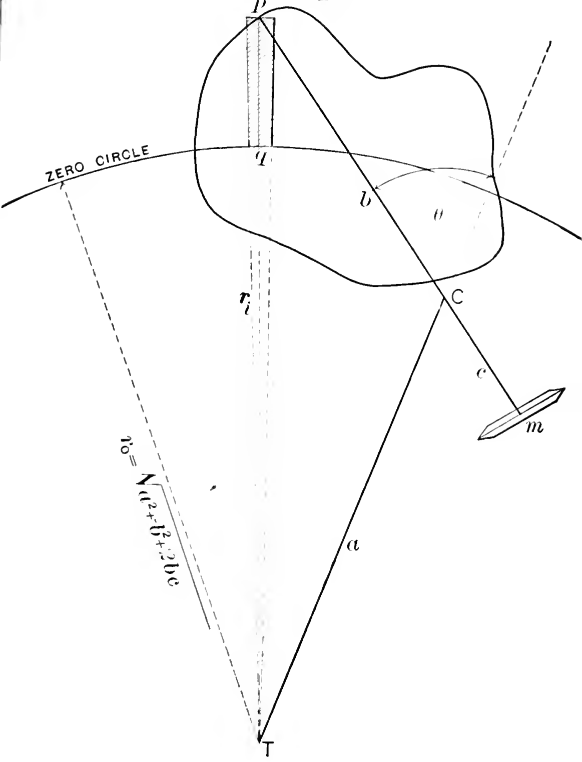
$Cp = \text{one portion of pole-arm} = b$;

$Cm = \text{the other portion of pole-arm} = c$;

$$\text{Area } pq = \tfrac{1}{2}w\,(a^2 + b^2 + 2ab\cos\theta) - \tfrac{1}{2}w\,(a^2 + b^2 + 2cb) = wb\,(a\cos\theta - c),$$

and, by a geometrical construction, the travel of the measuring roller is easily shown to be equal to the same expression. Mr. Brooks also explains why the area of the zero circle must be added to the reading if the figure to be integrated contains the center T. The following appears, however, to be a still simpler explanation. Referring to Fig. 7, it is evident that going around the outside of the zero circle corresponds to a movement taken continuously above the zero line OX when only the portion above OX is measured. In this latter case the curve could never be completely traversed as long as the pointer moves in one direction. Suppose, however, that the ends of OX are bent round and brought within the figure, then the motion in one direction will enable the complete circuit to be made; but only the portion outside the line, *i. e.*, corresponding to that originally above OX, will be measured by the roller, and that within must consequently be added to the recorded result. This quantity is evident-

Fig. 9



ly the area of the zero circle in the case of the Amsler planimeter, which must, therefore, always be added to the result when the center of the radius-arm is within the diagram to be measured.

As the Amsler planimeter alone, so far as the author is aware, has been modified to measure the area of any non-developable surface, this modification may be here noticed. The only surface of the kind to which it has been adapted is a spherical one. Fig. 10 shows the instrument, and from that figure it will be seen that the chief alteration is the placing of two joints *j j'*, one upon the radius-bar (*a*), and the other upon the pole-arm *b*, so as to allow the employment of the integrator for surfaces of varying curvature. The joints are equidistant from the end of each bar, and exactly opposite to each other—the radius-bar and pole-arm being now of equal length, and a pin *f* is placed on (*a*), which fits into a corresponding recess in (*b*), so that when the two arms are closed, they can be together bent at the joints

to the required amount, and, the joints being purposely made stiff, they will re-

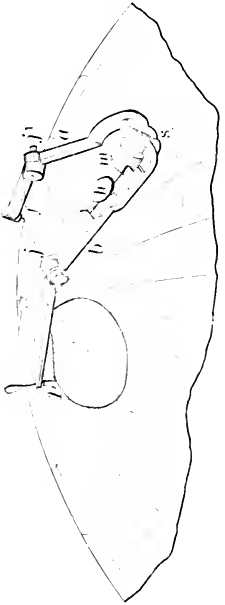


Fig. 10

main at the proper angle when the instrument is used. The joint (j) acts so that the tracing point (p) is always in the place of the axis of rotation of the

measuring roller. The theory of the action of this instrument has been fully explained by Professor Amsler, in an article in which the theory of the relations between measurement upon a spherical surface and upon a plane surface is discussed.

The various applications of the simple planimeter for finding areas are well known, and need not be explained; but there are some slight modifications of the instrument for special purposes, and one of these recently applied by Professor Amsler to his planimeter is worth noticing. This is illustrated at Fig. **11**, and is a device for reading at once the mean pressure given from an indicator diagram. Two points, U and V are seen, one (U) being upon the upper side of the bar A, which slides in the tube H, and one (V) upon the tube H itself. These points can be adjusted to the length of the diagram by inverting the instrument in the way shown in the figure, and the sliding-bar is then clamped by the screw S. This setting causes the reading of

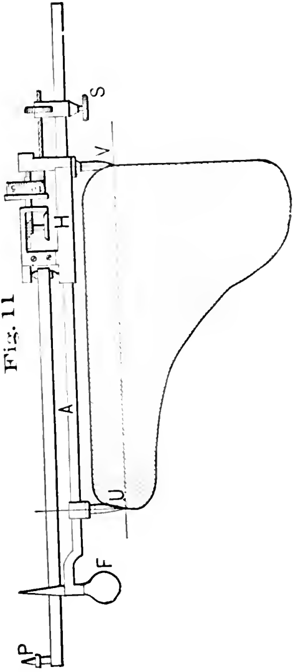


Fig. 11

the instrument, when the diagram is traversed by the pointer in the usual way, to give at once the mean height of the diagram in fortieths of an inch. The simple relation is as follows:

$$\frac{\text{Reading of measuring-roller}}{40} = \text{Mean height of diagram in inches.}$$

$$\text{Mean pressure} = \text{Mean height} \times \text{vertical scale of diagram.}$$

As an instance of the great saving of labor by the use of the Amsler planimeter, the author happens to know a civil engineer's office, where a large amount of earthwork quantities had to be taken out, the calculations proceeded slowly and with many repetitions, until one of the draftsmen procured a planimeter, and then the other, with the result of a great expedition of the work, and the almost complete absence of errors—and even then only in decimal places—where previously the divergence had been as much as by units.

Although the connection between the

disk and roller or linear planimeter and the polar planimeter has been shown, it is possible to regard them as acting upon different principles. The former may be considered as measuring the variation in the ordinate (y) by a change of effective radius of the circle on which the measuring-roller works, the latter measuring the same thing by a corresponding change in the sine of the angle which its plane of rotation makes, with its direction of motion over the surface on which it rolls. They have, in fact, been classified in this way as radius machines, and sine or cosine machines, for the slipping, although occurring in both, appears in the ordinary way of viewing the subject to affect the result in different ways. In the former, slipping is supposed to be entirely due to the variation in the value of (y), and only takes place when the ordinate changes in value: in the latter, the change is supposed to be effected by turning the pole-arm about its center, without any slipping at all. This distinction is, however, quite an imaginary

one, for it will be seen that if the curve be obliquely inclined in either case to the axis OX, the action of the measuring roller is precisely the same. Recently, a large number of what are called "Precision Polar Planimeters," have been designed and constructed, which combine in an obvious manner the above two principles of action, the disk giving motion to the roller, while the pole-arm carries it across the disk in an oblique direction. Thus, the advantages of a uniform and invariable surface of contact for the roller, and the convenience of the polar planimeter are combined, with the still more important advantage of a large relative turning of the measuring roller. Before describing a few different forms of the best of these instruments, the general theory upon which they work will be given; it will then not be difficult to understand the action of the several instruments without repeating the explanation in each case. It will be found that both the linear and polar planimeter are only special cases of application of the general

principle upon which the correctness of action of precision planimeters depends.

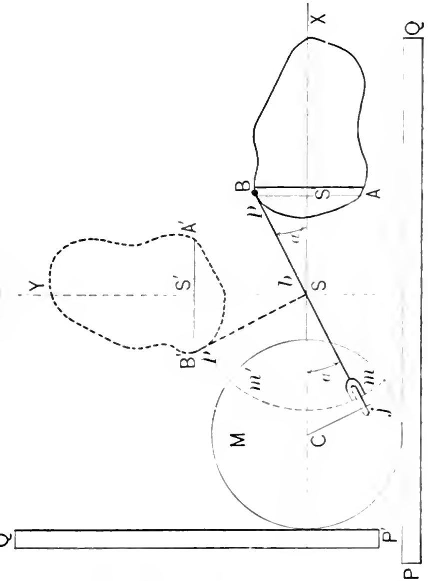


Fig. 12

It will be well to approach the matter from the same point of view as in explaining the linear planimeter. Let the disk M, Fig. 12, rotate about an axis C as it

rolls along the PQ, line parallel to OX, the pivot on the axle at C being attached to a frame which also carries another pivot S. This latter pivot always lies upon OX, and about it rotates a pole-arm b, carrying a pointer p at one end, and the measuring-roller m at the other end. The plane of rotation of the measuring-roller coincides with the direction of the pole-arm, and is carried over the disk in contact with it, along the arc mm'. Then from what was proved, p. 402, the motion of the roller m is exactly the same as if it were moved, so that its axis always coincides with Cj, the perpendicular upon the pole-arm from the center of the disk —provided only that its axis is always parallel to this line. Thus, adopting the previous notation, and taking

Sp=length of upper portion of pole-arm$=$R.

Then when the disk rolled through a distance $\triangle x$,

n_1=reading of roller m

y_1=ordinate SB.

$$\frac{\text{turning of roller}}{\text{distance turned by edge}}=\frac{2\pi r n_1}{\Delta x}=$$
$$=\frac{Cj}{CS}=\sin\alpha\,;$$

but $$\frac{S'B}{SB}=\frac{y}{R}=\sin\alpha\,;$$

therefore $$\frac{2\pi r n_1}{\Delta x}=\frac{y}{R},$$

or $$n_1=y\,\Delta x\times\frac{1}{2\pi r R},$$

which is the same result as in the case of both the linear and polar planimeters. In practice, the portion of the pole-arm which carries the pointer is usually perpendicular to the other portion, as shown by the dotted lines, Fig. 12. In this case, the direction of motion of the disk and frame carrying the center of the pole-arm S must be taken parallel to the guide P′Q′, that is, perpendicular to the former direction. It has already been shown in the case of the Amsler planimeter, that it does not matter in what path the center S of the pole-arm is carried, so long as the foregoing conditions are observed, and thus there are several forms of pre-

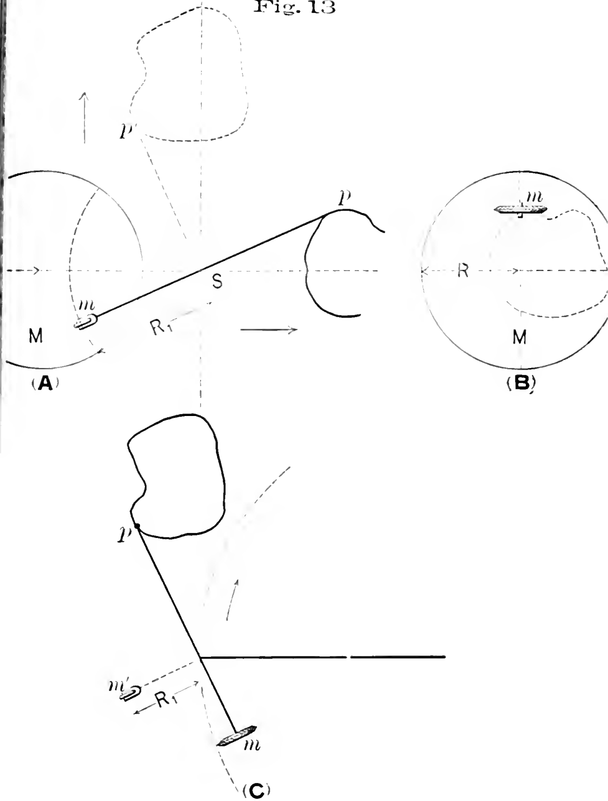
cision polar planimeters in which the point S is carried in the arc of a circle instead of along a straight line. It may now be made clear, from Fig. 13, that the first two kinds of planimeters are special cases of the last.

(A) Fig. 13. Let R be the radius of the disk, R_1 the radius about which the roller *m* is carried. Then the area of the diagram as already explained can be measured by either pole-arm S *p* or S *p′*.

(B) Fig. 13. Let the radius R_1 of the pole-arm become infinitely great, while R remains finite; thence *m* moves across the disk M in a straight line usually, but not necessarily, through the center, and the linear planimeter is the result.

(C) Fig. 13. Let the radius of the disk become infinitely great, and any motion of such an imaginary disk under these conditions would make the result equivalent to moving the roller over the surface of the paper. Therefore the disk may be removed, and the elementary form of polar planimeter is obtained, the roller being placed in either position

Fig. 13



as shown at m or m', without affecting the result.

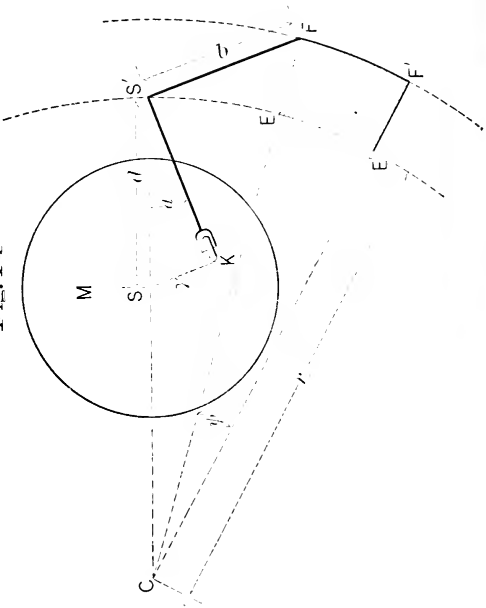


Fig. 14

The following is a simple explanation of the action of the precision polar planimeter

Let M (Fig. 14) be the disk, which can be turned by any suitable means through a distance corresponding to the linear travel of its center about C.

Let r_0=radius of zero circle (E′ES′).
r=radius of any circle FF′.
a= ∠ turned through by pole-arm, when the pointer moves from the zero circle to the circle FF′.
ϕ= ∠ turned through by radius arm, CS, when an element EE′ F′F is being described.
a=radius arm=CS′.
b=pole-arm=FS′.
d=SS′.

Then from the figure—

and $$r_0^2=a^2+b^2$$
$$r^2=a^2+b^2-2ab\cos(90+\alpha)$$
$$=a^2+b^2+2ab\sin\alpha.$$
Therefore $r^2=r_0^2+2ab\sin\alpha$,

or $$\sin\alpha=\frac{r^2-r_0^2}{2ab}$$

Now the turning of the plate is proportional to ϕ, and may, for the arc FF′, be taken as equal to $r_0\phi c$.

$$y=\mathrm{SK}=\mathrm{SS'}\sin\angle\mathrm{SS'K}=d\sin\alpha,$$

where c and d are constant quantities; also if ρ equal the radius of the roller.

then $\frac{y}{R}=\frac{\text{linear distance by edge of roller}}{\text{distance traveled by edge of disk}}$

$$=\frac{2\pi\rho n_1}{r_0\psi c};$$

$$\therefore y=\frac{2\pi R\rho n_1}{r_0\psi c};$$

Therefore $\frac{2\pi R\rho n_1}{r_0\psi c}=d\sin\alpha=\frac{d\times(r^2-r_0^2)}{2ab}$

or $\qquad n_1=\frac{r^2-r_0^2}{2}\psi\left(\frac{r_0 cd}{2\pi R\rho ab}\right);$

but $\frac{r_0 cd}{2\pi R\rho ab}$ is a constant quantity, and may be made equal to unity.

Therefore $n_1=\frac{r^2-r_0^2}{2}\psi=$ area of element EE′FF′.

Thus n_1 is a measure of an element of area, and as the motion of m due to the turning of the pole-arm in moving to a larger or smaller circle does not affect the reading when the pointer at F has passed round a closed curve, the final reading of the roller gives the area of any figure.

The actual construction of the precision polar planimeter appears to have

Fig. 15

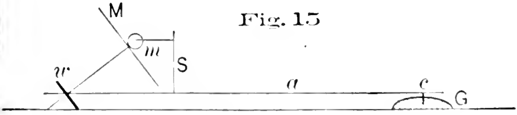
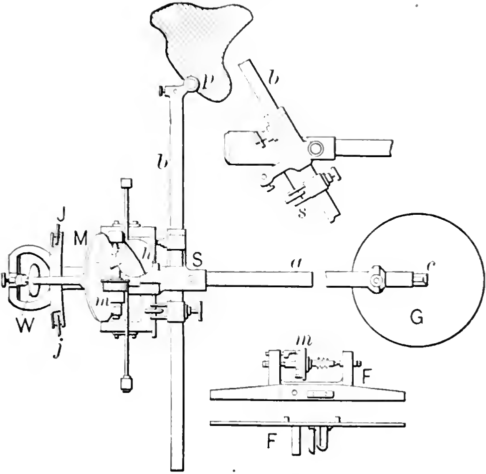


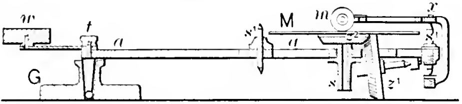
Fig. 16



been first carried out by Mr. Hohmann, Bauamtmann of Bamberg, in 1882, in conjunction with the well-known mechan-

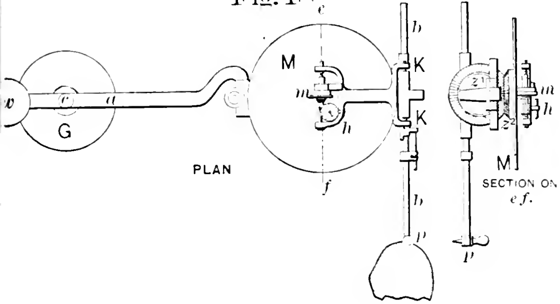
ician Mr. Coradi, of Zurich. A plan of the first instrument is given in Fig. 16; but it will be more easily understood by reference to the diagram, Fig. 15, which shows a frame (a) pivoted at one end (c) to a weight (G), about which it turns. This frame carries a small disk (w), which rolls in contact with the surface of the diagram, and gives motion to the disk M. The roller m is moved across the disk in the horizontal direction by a pole-arm centered on S as axis. Referring to Fig. 16, which is lettered in a similar way to Fig. 15, it will be seen how the pole-arm, in turning about the center S, effects this motion. A plan and elevation of the frame F, which carries the roller m, is shown on a larger scale, and this frame is moved backwards or forwards through a slot in the supporting frame. The roller m has the arrangement of the screw and worm for obtaining the readings of the dial h, as in the Amsler planimeter, and also the vernier in conjunction with the measuring roller. Two rollers, j j', serve to

Fig. 17



SECTIONAL ELEVATION

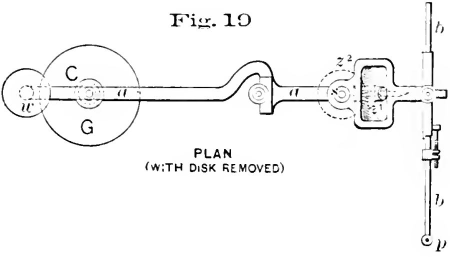
Fig. 18



PLAN

SECTION ON ef.

Fig. 19



PLAN
(WITH DISK REMOVED)

balance the instrument. The details of the arrangement by which the length of the pole-arm b is adjusted are also shown on a larger scale.

In this instrument, the fact that the disk is inclined at an angle makes no difference in the theory of its action, and as the roller W obviously drives the disk so that the angular motion corresponds with the angular motion of the radius bar a, the explanation already given makes its mode of operation clear. The case is rather simplified by the fact that the roller m is moved radially across the disk.

An instrument of similar kind has been designed and recently described by Professor Amsler-Laffon. This is shown in Figs. 17, 18 and 19, where it will be seen that this disk M, which is now horizontal, is turned by means of bevel-wheels z_1 z_2, the back of one of which forms a portion of a frustum of a cone rolling about the center C of the radius-arm a a. The center is itself a sphere, which allows any side motion of the instrument

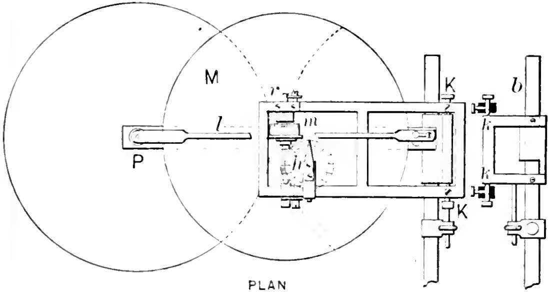
due to the inequality of the surface to take place without affecting the accuracy of the result. The necessary pressure cf the roller upon the disk is obtained by allowing the weight of the portion of the frame *b* which carries the roller *m* to rest upon the disk by being pivoted by the centers KK (Fig. 18). A peculiar feature of the instrument is that the pole-arm frame can be centered either within or without the frame. If placed in the former position, the reading is twice as great as in the latter, the positions of the centers being purposely adjusted to effect this. The frame can be taken off one center, *s′* (Fig. 17), by unscrewing a set screw at *x*, and at once placed upon the other. The weight *w* can be adjusted in any position by means of the nut and screw *t* (Fig. 17), and so the pressure of the pointer *p* upon the surface of the diagram may be regulated.

In both the above instruments the disks derive their motion from a roller in contact with the surface of the diagram, but in the next two instruments to

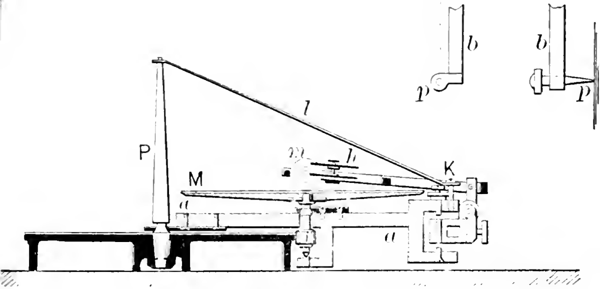
be described, Messrs. Hohmann and Coradi have caused the disk to be turned in a manner which prevents any such error as from the possible slipping of the above roller. The first instrument of the kind is shown (Figs. 20, 21) in plan and elevation. The disk M is carried by a frame (*aa*) as before, but the frame now swings about a circular stand, the edge of which is toothed, so that the pinion (*i*), which is upon the axis of the disk, is turned, and therefore the disk itself, with the same angular velocity. The weight of the frame and disk is, to a great extent, taken off by means of the light rod (*l*), which swings about a central pillar **P.** A side view of the pole-arm is shown, and the mode of adjusting it and supporting the portion which carries the roller (*m*), so that by means of centers KK the weight of that portion of the frame is allowed to rest upon the disk.

It is evident that this instrument works upon identically the same principles as the foregoing ones. This "Freely swinging" precision planimeter was fol-

Figs. 20 and 21



PLAN



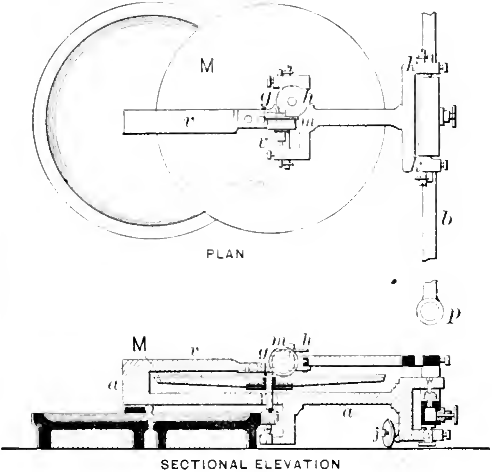
SECTIONAL ELEVATION

lowed by another, called the "Plate" planimeter (Fig. 22), which is of still simpler construction. In this form advantage is taken of the fact that the measuring roller need not have its path through the center of the disk, and a support (v) is obtained above the disk, so that its pivot (q) can work between centers, the weight of the frame being supported by rollers (j). The portion of the pole-arm which carries the roller (m) is (as in the last case) pivoted between the centers KK. The dial for higher readings is as in the case of the previous instruments denoted by h.

The last and most recent modification is the "Rolling planimeter," of Coradi. This approaches nearest to the diagram (Fig. 12), which completely explains its action. Here the center of the radius-arm is removed to an infinite distance, and the center of the disk and that of the pole arm are carried along straight lines parallel to the axis OY in that figure. The way in which this is effected is seen from Figs. 23 and 24, which show Co-

radi's rolling planimeter in plan and elevation. Two rollers (cc') are in contact with the surface of the diagram, in

Fig. 22



their axis is a bevel wheel (z_1) (Fig. 24), which gears with another bevel-wheel (z_2), which is upon the axis of the disk. Thus the wheels z_1 z_2 are turned as the frame is rolled along, and, consequently, the

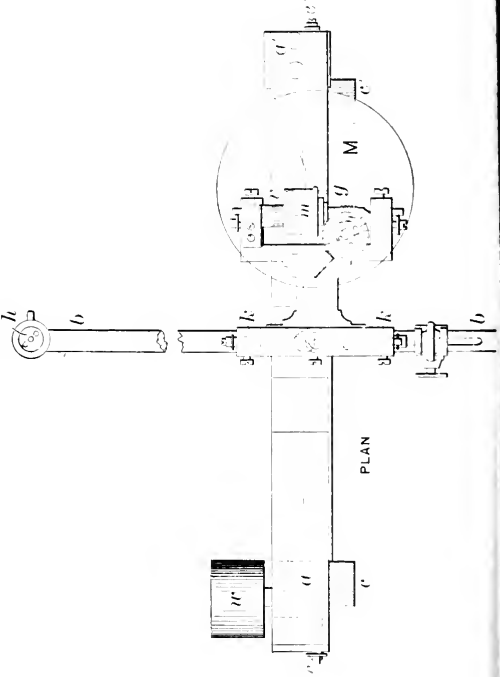
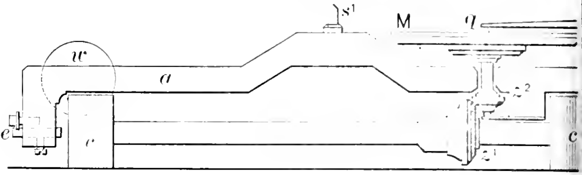


Fig. 23

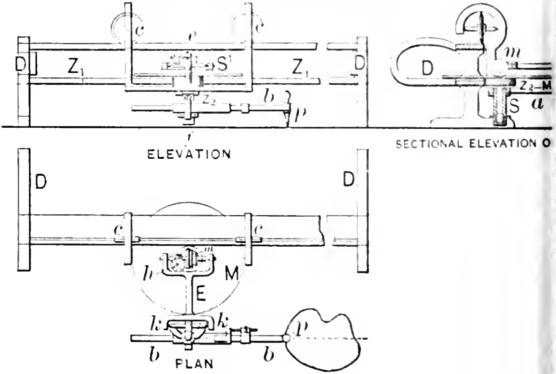
disk itself. The axis of the rollers *cc* works upon the centers *ee*, which are set-screws in the frame (*aa*). The disk M is also carried between centers (*qq*), as in the instrument last described, and, also, as in that case, the path of the roller does not pass through the center of the disk. This instrument, which has many advantages, and, notwithstanding that it rolls on the diagram surface, gives results of great accuracy, has been examined with great care by Professor Lorber, who has given a lengthy description of it and a full account of its theory.

The last planimeter of this kind to be examined is one by Professor Amsler. This instrument, shown in Figs. 25, 26, 27, differs from the last in that the tooth-wheel z_2 works in gear with a rack z_1 z_1, which is cut upon a fixed frame DD. Thus, although it is supported by the rollers *cc*, there is no possibility of slipping as far as the turning of the disk is concerned. The rollers run in a groove cut in the frame DD, and the action of the instrument is easy and smooth. The

Fig. 24



Figs. 25, 26 and 27



theory of its action is identical with that of the foregoing one, as explained by

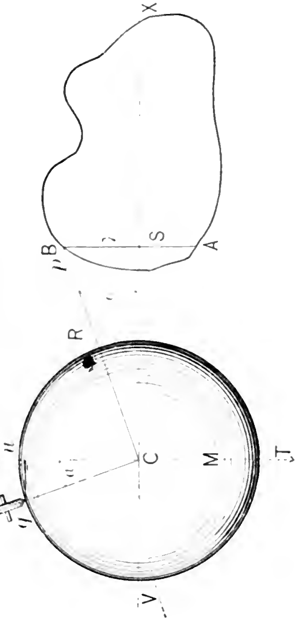


Fig. 27

means of the diagram (Fig. 12). The various parts are lettered in the figures to correspond with the explanations of that instrument previously given.

In the instruments hitherto described the surfaces of revolution are limited to the disk and cone, but various other surfaces may be made to replace these. The only one that has been so employed is that of the sphere; and in the present class of instruments, in which slipping takes place, the following property of the sphere is made use of: Let a sphere M (Fig. 28), which replaces the disk (Fig. 2), roll along the axis OX. Then suppose the roller m can, by suitable means, be moved round the surface so that its plane of rotation shall always contain the center of the sphere and be perpendicular to the arm CB, which corresponds to the pole-arm of the former instruments; it is evident that if the perpendicular be drawn from q, the point of contact of m with the sphere, to CZ the axis of rotation of the sphere, meeting it in the point u, the line qu is the radius of the rolling circle of contact of the measuring roller. Therefore $\dfrac{\text{motion of measuring roller}}{\text{motion of sphere along OX}} =$

$$=\frac{qu}{\mathrm{CV}}=\frac{qu}{q\mathrm{C}}=\sin\,a.$$

But from the figure $\frac{SB}{CB}=\frac{y}{R}=\sin a.$

Therefore, adopting the same notation as hitherto used,

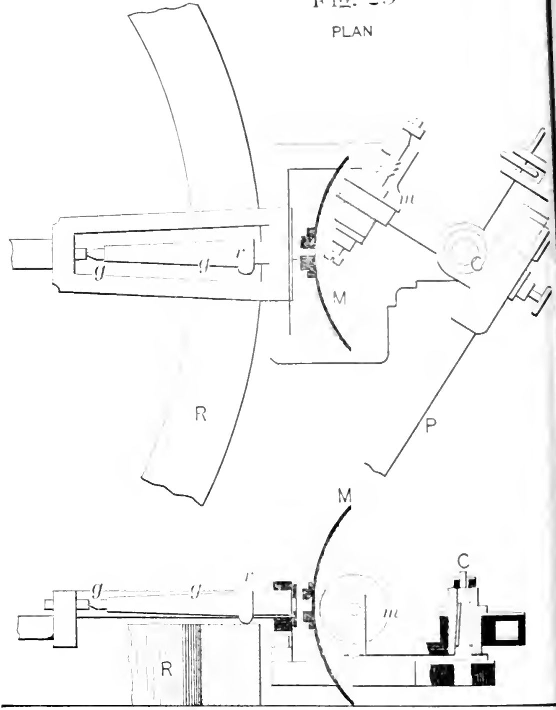
$$\frac{\text{motion of } m}{\text{motion of } M}=\frac{2\pi r n_1}{\triangle x}=\frac{y}{R}$$

$$n_1=y\triangle x\times\frac{1}{2\pi r R},$$

which proves that the area of the curve may be measured by any device, on the principle of Fig. 28. It may be shown in the same way as on p. 18, that the result is similar if the sphere rolls upon the arc of a circle, about any center as T, instead of along the straight line OX. Planimeters of this kind have been constructed by Mr. Hohmann and Professor Amsler. In both cases only portions of the whole surface of a sphere have been employed, and the motion is given by means of an axis, instead of by rolling the spherical surface upon the diagram. In Mr. Hohmann's planimeter, shown in plan and elevation, Fig. 29, the concave surface M is used. Rotation is given to

Fig. 29

PLAN



SECTIONAL ELEVATION

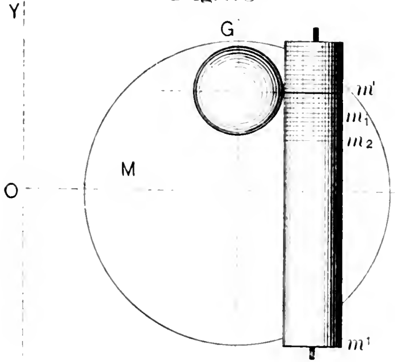
this by means of an axis gg, an enlarged portion of which (r) rolls upon a circular metal path R. The pole-arm P turns about a center c, and so causes the rolling circle of the measuring roller m to vary according to the foregoing principles. This instrument has not come into use. Professor Amsler has employed the convex surface in an instrument somewhat similar to the one described, except that better provision is made for obtaining the required pressure between the surface of the roller and sphere, and for giving rotation from the roller-path.

Planimeters in which only Pure Rolling Motion is Assumed to take Place.

There have been many efforts to design instruments in which no slipping shall take place. These efforts have resulted in the production of various instruments which, though they differ in external form and mechanical action, yet rely upon the same mathematical principle of action as the planimeters already dealt with, the particular form of disk and roller, or sphere and roller, being taken. Thus, in every case there is a measuring roller, or its equivalent, the rate of motion of which has to be varied by some means or other. It is in the method by which this is done that this class of planimeters differs from the other. Instead of obtaining the variation of the measuring roller in bringing it into contact with circles of different linear velocity by sliding it over the surface of the disk or sphere, one or other

of the two following principles are employed. A device equivalent either to (1) bringing in succession a series of measuring rollers into contact with the different imaginary circles; or (2) bring-

Fig. 30

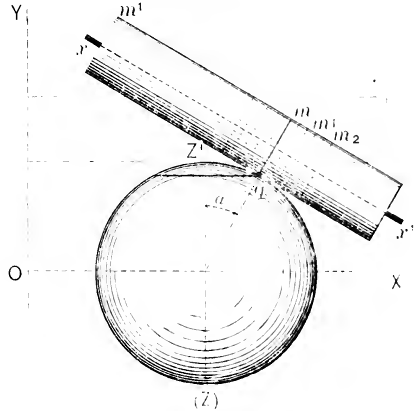


ing circles of different linear velocity into contact with a single fixed measuring roller.

The disk-globe and cylinder-integrator of Professor James Thomson belongs to the former class. In this a sphere G (Fig. 30) rolls over the surface of the disk

M, but is also in contact with a cylinder *mm'*. The motion of G in direction OY is that in which the roller would slip in the ordinary disk and roller, and does not

Fig. 31



affect the motion of rotation of *mm'*. On the other hand, the motion in direction OX, which is due to the turning of the disk, is entirely imparted to *mm'*. Thus, as G rolls along *mm'*, the same effect is, in theory, produced as if a series of roll-

ers m, m_1, m_2, etc., upon the same axis as the cylinder, were successively applied to the surface of the disk, and all slipping, at any rate from this cause, is avoided. The actual mechanism which has not been employed for a planimeter takes a slightly different external form in the harmonic analyzer of Sir W. Thomson's tide-calculating machine.

The devices which have now to be considered as solutions of the problem under consideration by the first method, are used in connection with the geometrical property of the sphere already discussed, and upon one similar to it.

Let M (Fig. 31) be the plan of a sphere rolling along the line OX, carrying with it, by a frame not shown, a cylinder (m) which can roll about it so as to come into contact at any point q upon its horizontal great circle. Then the rotation of the cylinder may be employed exactly in the same way as the rotation of the roller on the integrator described on page 416, and shown by Fig. 28; but in the present case, instead of causing the roller to

slip over the surface, the rolling of the cylinder is practically equivalent to bringing in succession a series of rollers m, m_1, m_2, etc., upon one axis in contact with it.

This principle has been employed both by Professor Mitchelson of Cleveland, U. S., and Professor Amsler. The mechanism of Professor Mitchelson's instrument is shown in Fig. 32. In this a flexible steel band or chain F, passing round a semi-circular arc D, forces the cylinder C to roll on the sphere G. The cylinder is carried by a frame E, which slides along the bar A, by which it is supported. The mode in which it is proposed to apply it to the ordinary Amsler planimeter is shown on a smaller scale, Fig. 32A, where b is the pole-arm, a the radius bar, and t the center of rotation of the latter.

Professor Amsler's planimeter on this principle is similar to the foregoing, except that instead of being carried by two guides as sleeves by a bar, the cylinder frame is supported on rollers from a

Fig. 32

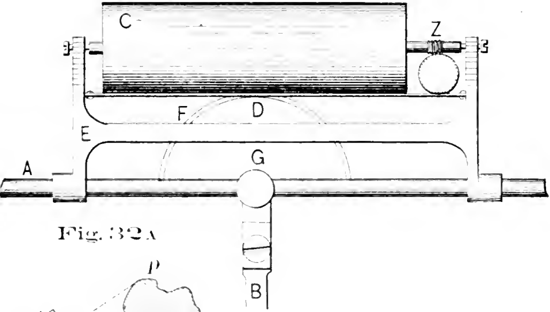
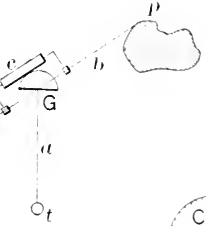
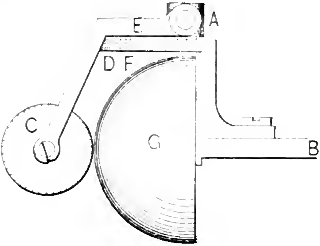


Fig. 32A





frame above, the rolling friction on the latter being less than that of the cylinder on the sphere. Thus, the cylinder always moves to its required position. The motion of the spherical surface is obtained from a bevel-wheel upon its axis, which gears with a larger one formed upon the edge of a circular stand or support of the instrument.

A similar principle of the geometry of the sphere has also been employed in an instrument suggested in a paper in 1855 by the late Professor Clerk Maxwell, when an undergraduate at Cambridge. Instead of the cylinder in Fig. 31, let a sphere m' roll around on the sphere (M), as shown in Fig. 33. Then, from the property of the sphere, which is proved at length in the above paper, the turning of the sphere m' about its axis of rotation xx, relatively to the turning of M along OX, is proportional to the tangent of the angle α in the figure. In the other case it will be remembered that the turning of the cylinder or disk was proportional to the sine of the same angle. By

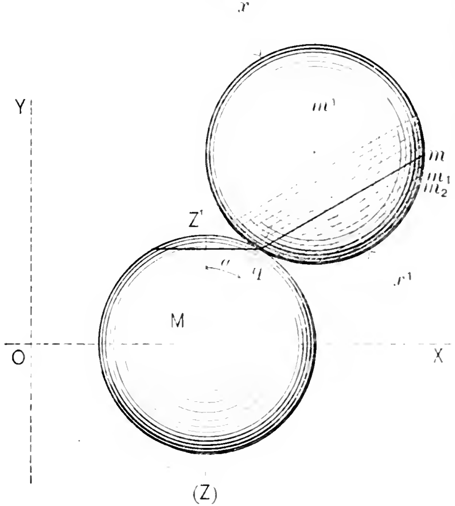
suitable means the principle can be employed in the construction of a planimeter. Two forms of such planimeter are shown in the paper, and though they are both in the form of the linear planimeter, and are scarcely suitable for practical application, yet the matter is dealt with in a way worthy of the inventor. It is evident that this is another case of bringing the equivalent of a series of rollers m, m_1, m_2, into contact with the sphere, though these are no longer of one size, but vary from a diameter zero to a diameter of the size of that of the sphere m.

Coming now to the instruments in which the alternative device adopted for the avoidance of slipping is by bringing into contact with one roller different circles of the disk M, or of its equivalent.

This may be done in the following way: Instead of allowing the cylinder (m) to roll on the sphere M (Fig. 31), and so to change the radius of the imaginary rolling circle (whose diameter is qq') on which it rolls, suppose that the

cylinder is kept in contact as shown by the dotted lines, and the axis of rotation zz' of the sphere is turned, as, for in-

Fig. 33



stance, would happen if a sphere in combination with rollers were used as suggested for an anemometer by Mr. Ventosa, through an equivalent amount, *i. e.*, through the angle a. This will give the

same result as far as the rotation of the cylinder is concerned, but with an important difference. The cylinder (in Fig. 31) or sphere (in Fig. 33) is no longer needed, and may be replaced by the original measuring roller, whose axis has a fixed position parallel to OX. It will be seen that this device practically amounts to bringing different circles on the sphere **M** into contact with the measuring roller (m), with the great advantage that exactly the same circle on the sphere M is scarcely likely to again roll in contact with the roller (m), though of course the radius may be the same. This method has been recently proposed by the author, and the mode of carrying it out without involving slipping, by what is called the "sphere and roller mechanism," which mechanism has been explained and developed at length in a paper before the Royal Society. It need here be only remarked that the planimeter there described, and afterwards exhibited to the British Association at Montreal, was of the linear form, and of little practical

use; but the author has since completed a polar planimeter and exhibited it before the Royal Society.

But one more area planimeter remains to be mentioned, and this is the one invented and brought before the Physical Society by Mr. C. V. Boys. The principle of action is briefly this: A wheel or roller, which is not supposed to slip sideways on the diagram, has its plane of rotation kept always at an angle a to the axis OX of the figure to be integrated, such that

$$y = \text{ordinate of the curve}$$
$$= \tan a \times K$$

where K is a constant, and y is the ordinate with respect to OX of that point on the curve which the pointer of the instrument is at the same instant tracing. If the component of a small motion of the wheel parallel to OX is $\triangle x$, and the component of the same movement parallel to OY is $\triangle t$.

Then $$\frac{\triangle t}{\triangle x} = \tan a = \frac{y}{K}$$

$$\triangle t = y \triangle x \times \frac{1}{K},$$

or the distance moved by the wheel parallel to the axis OY becomes the measure of an element of area. It is easy to see that the height moved by the wheel becomes a direct measure of the area of the figure. Various examples of the action of this planimeter, called by the inventor the tangent integrator, are given by Mr. Boys; but the action is obviously limited, and an investigation of the theory reveals the fact that it is only a special case of the general problem, not only of the method of applying circles of varying diameter to one roller, but of the sphere and roller mechanism itself. This will be rendered clearer by stating that, in order to employ the component parallel to OY, the roller was made to work against a cylinder, which, by its turning, acted as the measuring roller. Evidently the length of the cylinder limited the travel in that direction. The cylinder was carried bodily along in the direction of its axis (corresponding to OX), and made to effect its own turning, the amount of turning varying with the tan-

gent of inclination of the wheel, and this was sufficient in the application to the steam-engine integrator to be hereafter described, where the longitudinal motion of the cylinder could be made proportional to the stroke. Mr. Boys endeavored, by various means, to obtain continuous motion in both directions, one being equivalent to bending the ends of the cylinder round, and so attempting to solve the difficulty by what he has termed a "mechanical smoke ring." The author, however, by approaching the matter from a different point of view, designed the sphere and roller integrator, which is nothing more or less than the inversion of the mechanism of Mr. Boys. In this the roller of Mr. Boys is replaced by the sphere, and instead of the two motions, one of the cylinder about its axis, and one of the cylinder longitudinally, the two rollers are used. It may be easily shown that the turning of the plane of rotation of the roller of the tangent integrator is equivalent to changing the axis of the sphere in the sphere and roller integrator.

Moment Planimeters.

The moment of an area, and its moment of inertia about a given line, may be obtained mechanically upon similar principles to those by which a simple area was obtained. If ABCDE, Fig. 1, be the figure whose moment of area and moment of inertia are required about any line OX; then, taking any element of area AB, if y=height of upper portion SB, then the moment of area of the element SB about OX

is $$m = \text{area of SB} \times \frac{y_1}{2}$$

$$= \tfrac{1}{2} y_1^2 \triangle x.$$

Similarly, the moment of inertia of the element is

$$i = \tfrac{1}{3} y_1^3 \triangle x.$$

The sum of an infinite number of such expressions as these, when $\triangle x$ become infinitely small, gives respectively the moment of area and the moment of

inertia of the whole figure according to the expressions.

$$\text{Moment of area} = M = \tfrac{1}{2}\int y^2 dx,$$
$$\text{Moment of inertia} = I = \tfrac{1}{3}\int y^3 dx.$$

Now, there are two possible ways of obtaining these results mechanically. One of these ways is by applying for the purpose the suggestion made by Sir William Thomson in connection with the disk globe and cylinder integrator of Professor James Thomson, of using a train of such mechanisms to obtain the integration of a simple linear differential equation. By certain simple arrangements,

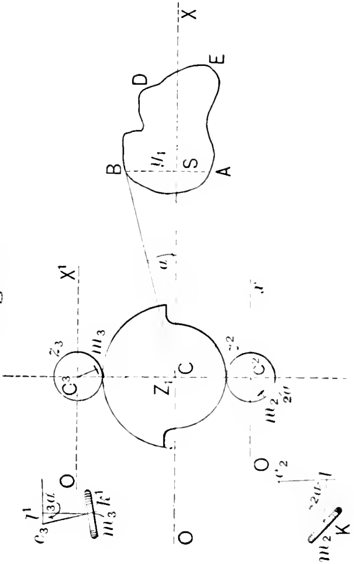
The first mechanism would give $\int y dx$,
" second " " " $\int y^2 dx$.
" third " " " $\int y^3 dx$.

This method need not be further considered here, since, so far as the author is aware, it has never been carried into actual practice. It may be, however, said that the mechanical difficulties in the way of causing the measuring wheel or roller of the first mechanism to actuate the second, and the roller of the second to actu-

ate the third, without introducing serious error, are not easy to overcome, and require a very easily working piece of apparatus. The author has discussed the applications of the sphere and integrator for the purpose, in a paper to the Royal Society.

The other principle is to cause the measuring roller to be directly turned at a rate which is made to vary, not as in the simple planimeter with the value of the ordinate (y), but with its second or third power. Though no method of directly doing this has apparently yet been suggested, yet the same result is practically effected by the beautiful application of a mathematical principle in the "moment integrator" of Professor Amsler.

Let the pole-arm CB (Fig. 34) be attached to a toothed segment (z_1), one portion of which gears with a toothed wheel z_2, the radius being as 2 to 1. Let the center C of z be carried along OX, while the center of C_2 of z_2 is carried along a line ox parallel to OX. Let m_2



O
X¹
C³
z₃
m₃
c₃
l¹
3α
m₃ k¹
O
Z₁
C
α
B
D
y₁
S
X
A
E
z²
C²
m₂
2α
x
O
c₂
2α
l
m₂
K

Fig. 34.

be a roller acting in every way as the measuring roller of the Amsler planimeter, whose axis is carried in the plane of the wheel z_2, its direction passing through the center C_2. When the pole-arm coincides with OX, let the plane of rotation of the roller m_2 be parallel to OX, and its axis parallel to OY. When the pole-arm is turned through an angle SCB$=\alpha$, the angular motion of the wheel z_2 is twice that of the arm; thus the roller m_2 takes the position shown in the figure.

This is so because $\dfrac{\angle^r \text{ motion of } z_2}{\angle^r \text{ motion of } x_1}=$

$$\frac{\text{radius } z_1}{\text{radius } z_2}=\frac{2}{1},$$

$$\therefore \angle Kc_2l=2a.$$

Suppose the pointer p to move through the width of the element SB at a height $=y$, and with it z, and z_2, the roller m_2 being in contact with the diagram surface. Then, by what was proved in the case of the Amsler planimeter, and adopting the same notation.

$$\frac{\text{Turning of } m_2}{\text{Motion of translation of } m_2}=\frac{2\pi r n_2}{\triangle x}=\frac{lc_2}{c_2 K},$$
$$=\cos 2a$$
$$=1-2\sin^2 a,$$

but $\frac{SB}{CB}=\frac{y_1}{R_1}=\sin a$ (where $CB=R_1$),

$$\therefore \frac{2\pi r n_2}{\triangle x}=1-2\sin^2 a=1-\frac{2}{R_1^{\,2}}y_1^{\,2},$$

or $$n_2=\left(\frac{1}{2\pi r}\right)\triangle x-\left(\frac{2}{2\pi r R_1^{\,2}}\right)y^2\triangle x.$$

When the complete travel of the curve has been made, the sum of a series of quantities similar to the first becomes zero; so that, by making the constant $\left(\frac{1}{\pi r R_1^{\,2}}\right)$ equal $\frac{1}{2}$, the reading of the roller gives the value—

$$M=\tfrac{1}{2}\int y\,dx,$$

or the moment of area of the figure BDEA.

For the moment of inertia the segment of z_1 is used, the radius of which is three times that of another wheel z_3, with which it gears. The action of a roller m_3, carried by the wheel z_3, is exactly the same as that of m_2, except that its angu-

lar motion is three times as great as the pole-arm CB, instead of twice as great, as in the case of the other roller.

By reasoning similar to that already adopted, and taking the plane of rotation of m_3 perpendicular to OX in its initial position, instead of, as in the former case, parallel to it—

$$\frac{\text{travel of } m_3}{\text{motion of translation of } m_3}=\frac{2\pi r n_3}{\triangle x}=\frac{l' c_3}{c_3 k^1}$$
$$=\sin 3a$$
$$=3 \sin a-4 \sin^3 a.$$
$$\frac{SB}{CB}=\frac{y_1}{R}=\sin a,$$

Therefore $\frac{2\pi r n_3}{\triangle x}=3 \sin a-4 \sin^3 a=$

$$=3\frac{y_1}{R_1}-4\frac{y_1^{\,3}}{R_1^{\,3}},$$

Or $n_3=\left(\frac{3}{2\pi r R_1}\right)y_1 \triangle x-\left(\frac{4}{2\pi r R_1^{\,3}}\right)y_1^{\,3} \triangle x,$

which, when the pointer is taken around the curve, gives, with suitable values of the constants,

$$n_3=\int y dx-\tfrac{1}{3}\int y^3 dx$$
$=$area of BDEA—moment of inertia of BDEA

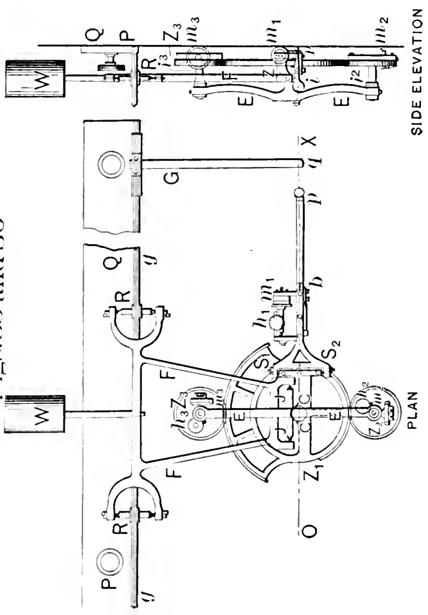
$$= \quad A \quad -I.$$

or $\quad I= \quad A \quad -n_3.$

The instrument, Figs. 35, 36, has an area planimeter attached to it, so that, by reading the rollers m_1 and m_3, and subtracting the results, the moment of inertia is obtained.

The details of the moment planimeter shown (Figs. 35 and 36) are easily explained. A guide PQ of steel has a groove *gg*, which is placed parallel to the axis OX by means of the gauges G, one, as shown, being at each end, which are adjusted with their points q upon the line OX. The rollers RR run in the grooves *gg*, and support a frame FF, which carries, by means of an axle JJ, the frame EE. This frame supports the toothed segment z_1, and the two toothed wheels z_2, z_3, upon vertical axis. The former between centers, one of which is shown, Fig. 36, i, the latter by steel axles within the column $i_2 i_3$. The pole-arm carries, in addition to the pointer p at the end, the roller m_1 with its dial h_1, forming an ordinary planimeter, and is itself carried on the centers $s_1 s_2$. The two other rollers and dials are shown as

Figs. 35 and 36



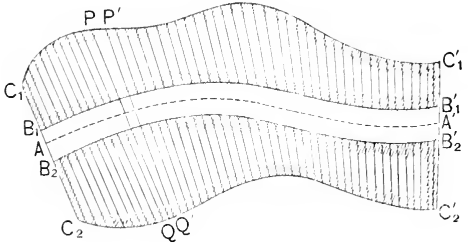
m_2m_3 and h_2h_3 respectively. The weight W serves to balance the instrument, so as to avoid undue pressure on the paper, and the motion is so smooth as to enable a curve to be traced with the greatest ease and accuracy.

Attention has been called by the late Dr. Merrifield and others to the valuable applications of this instrument for purposes of naval architecture, but so far as the author is aware, no account has been given in this country of its applications in civil engineering, as proposed by Professor Amsler. The following brief account of the methods in the case of calculating the contents of embankments, cuttings, etc., is therefore given from an abstract for which the author is indebted to the kindness of Dr. A. Amsler:

Let Fig. 37 be the plan of a portion of an embankment or cutting, the character of which is supposed to be the same throughout, viz., of uniform width of roadway, and uniform side-slopes, the surface of the ground, the gradient, and the horizontal curvature of the roadway,

being restricted in no way. AA' represents the center line of the railway; B_1B_1' and B_2B_2' its two borders; C_1C_1' and C_2C_2' the intersections of the side-slopes with the surface of the ground. Suppose now the embankment or cutting

Fig. 37



to be divided into thin layers by vertical planes, perpendicular to the center line AA' of the roadway; PQ and P_1Q_1 may be the intersections of two adjacent planes with the plane of the diagram.

Then if p = area of section PQ,
$\triangle s$ = interval between PQ and and P_1Q_1 measured upon the center of gravity of the section.

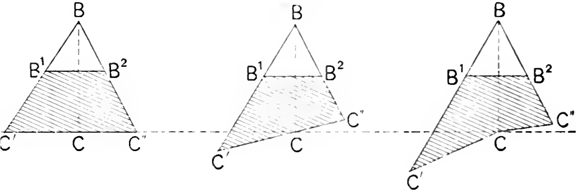
Total volume of embankment is (from one of the properties of Guldinus)—

$$V = \int pds,$$

the integral extending over the whole length of the embankment under consideration.

There are three cases dealt with in the

Fig. 38

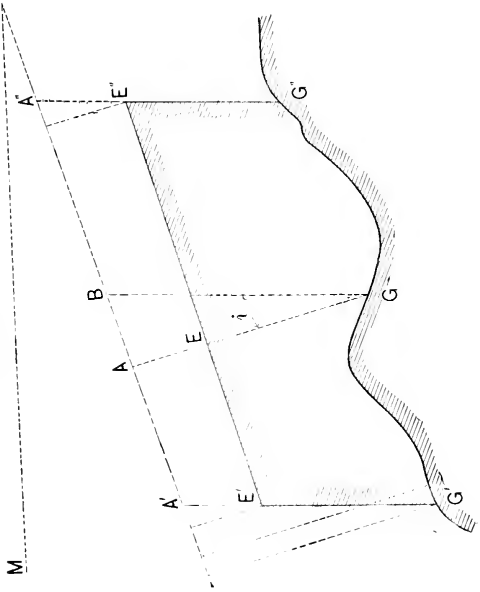


Paper of Professor Amsler, corresponding to the three forms of sections, I, II, or III, Fig. 38.

The first of these, I, is simple enough, since the center of gravity of the section always coincides in plan with the center line of the roadway, and the plan of operation is as follows:

Let Fig. 39 represent a longitudinal section of a portion of the embankment of uniform gradient, developed into a

Fig. 39



plane; the straight line E′EE″ represents the top of the embankment; G′GG″ the profile of the ground: the straight line A′AA″, which is parallel to E′EE″, is the locus of the imaginary vertex of the trapezoidal cross-sections. The level line MN is the line to which the offsets of the profile of the surface of the ground refer. BG shows the intersection of a vertical cross-section with the figure, and AG the intersection of a plane perpendicular to the top of the embankment (and also to the line A′AA″) with the figure.

Let $i = \angle AGB = \angle MNA =$ gradient;
$y = AG$ = distance of vertex to bottom of embankment;
$y_0 = AE$ = distance of vertex to top of embankment;
$2\beta =$ angle at vertex at A.

It may be easily proved the area of the section made by the plane AEG is—

$$p = (AG^2 - AE^2) \tan \beta = (y^2 - y_0^2) \tan \beta$$

but since $\quad V = \int p\,dx$

Therefore volume $= \tan \beta \int (y^2 - y_0^2)\, dx.$

And thus, if β is known, the volume of the portion E′E″ G″G′ (Fig. 39) is easily found with the mechanical integrator, thus:

Take A′AA″ as the axis of moments, and adjust the rail of the instrument so as to be parallel to it. Start the pointer anywhere on the shaded figure, and trace round it; the travel of the roller m_2 being denoted by M, the scale of the drawing longitudinally being:

$1'' = m$ feet,

and vertically $\quad 1'' = n$ feet;

then $\quad$ volume $= V = 20\ mn^2 \tan \beta \times M$ cubic feet.

It only remains to insert a known value for $\tan \beta$, which is easily done, thus:

Let Fig. 40 be a perspective view of the sections AEG and BG (Fig. 39), where:

$$\angle C_1BC_2 = 2\ \alpha.$$

Then from the diagram $\dfrac{C_1G}{AG} = \tan\ ;$

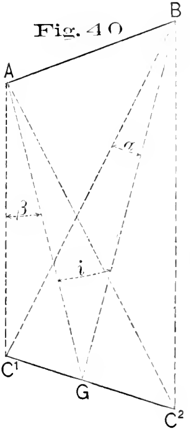
also $\frac{C_1G}{BG}=\tan\alpha$ and $\frac{AG}{BG}=\cos i.$

Therefore $$\tan\beta=\frac{\tan\alpha}{\cos i},$$

or $$V=\left(20\, mn^2\frac{\tan\alpha}{\cos i}\right)M,$$

where $\angle\alpha$ and $\angle i$ are known constants.

Fig. 40



To complete the calculations for the whole route separate portions are taken, with the various proposed gradients.

The above formula is exact for the integrator shown in Fig. 36, as arranged for English measures, a complete revolution of the measuring roller being taken as a unit of reading.

It is to be noted that nothing is supposed as to the curvature of the center line of the roadway horizontally, as it is supposed to be developed in

the figure. Also, that the aggregate error arising from the assumptions that the cross-sections are exact trapezoids will in most cases be very slight, on account of the errors in cuttings and those in embankments partly compensating for each other, in addition to the cutting and filling in each section, as shown in Fig. 41, where the small triangular portion in dotted lines C′DH represents the amount taken off the former, and added to the latter.

Alterations of the proposed roadway, otherwise involving tedious calculations, simply necessitate an alteration in the line A′AA″, and a repetition of the mechanical work of the integrator, but need no fresh diagram. In preparing the drawing, allowance should be made for ditches along the roadway in cuttings, which is easily done, as shown in Fig. 42, where B_1B_2, which equalizes the amounts taken and left, must be considered as the roadway line. In the case shown in Fig. 43, the excess of the embanking over the cutting is approximately equal to the

Fig. 41

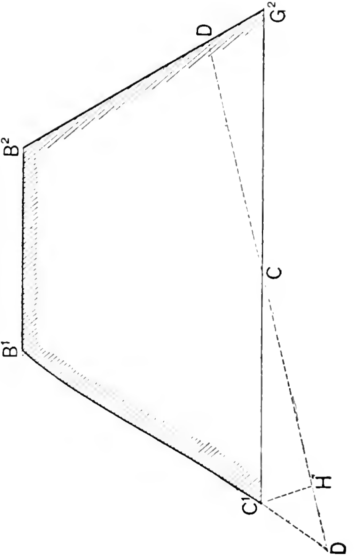
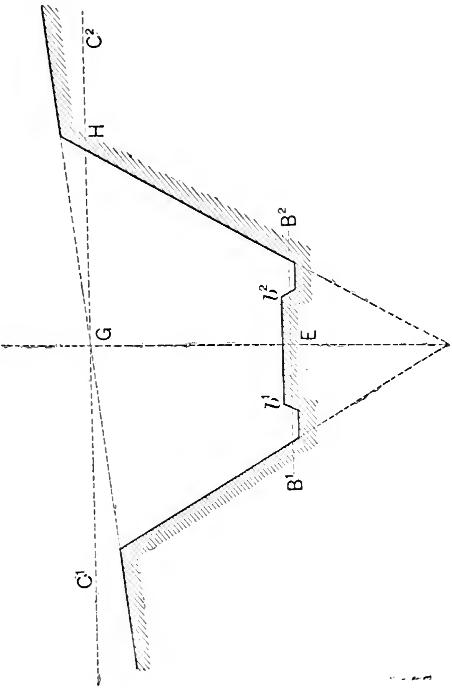


Fig. 42



layer above the dotted line C_1C_2. The contents of this layer could be measured either by considering it as an embankment, and treating it as such, or by the simpler—and for a first estimate sufficiently accurate method—of assuming its section to be a parallelogram. The area of the shaded portion (Fig. 43) is then simply to be measured, and the result, multiplied by the length of the road, gives the required contents. The supposition that the slopes CD and C′D′ are the same is also sufficiently accurate.

The foregoing is the first method described by Professor Amsler, and is extremely simple, but obviously only approximately accurate. The two other methods are capable of giving very accurate results, and are dealt with by him at considerable length. Only a short account of them will be given here.

The first thing to be noted is that, as a rule, the center of gravity of the section will not really coincide in plan with the center line of the roadway, but will curve at the line SPP′S′, Fig. 44, AA′ being the

Fig. 43

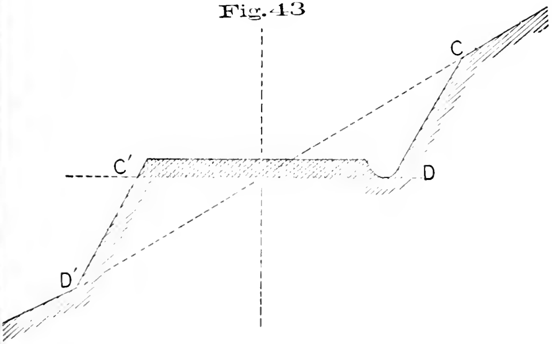
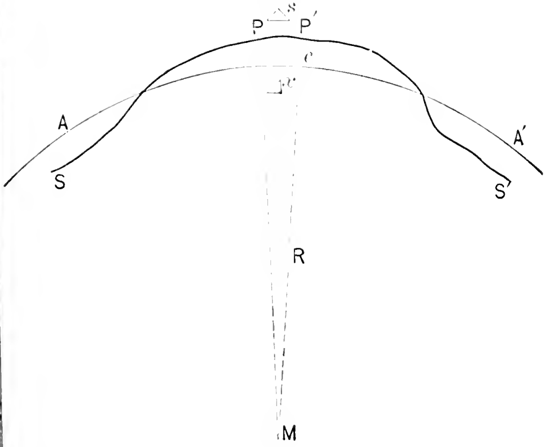


Fig. 44



true center line. Thus, in the expression $\int pds$, the value of ds does not coincide with dx, as hitherto assumed. From the figure it is seen that:

$$\frac{\Delta s}{\Delta x}=\frac{R+e}{R},$$

where R is the radius of curvature of the center line.

Therefore $$\Delta s=\Delta x 1+\frac{e}{R}),$$

or $$V=\int pds=\int pdx+\int\frac{pedx}{R},$$

and this expression must be used.

The first of the two methods assumes the base of section to be inclined, but not broken (Fig. 38, II), and the side-slopes, gradient, and radius of a given portion to be constant. A diagram is prepared, as shown in Fig. 45, in which the dotted lines now represent intersection of the sides of embankment with the surface of the ground, which do not, as before, coincide with the contour of the center line.

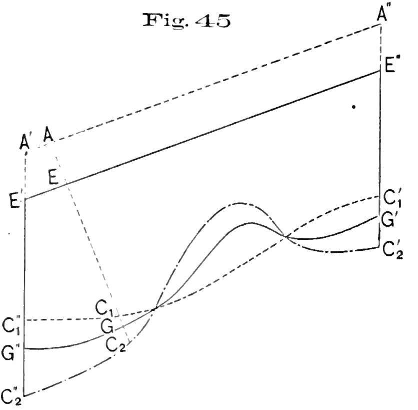
From this figure $y_0 = AE$

$y_1 = AC_1$

$y_2 = AC_2$

$2\beta = \angle$ at vertex A.

Fig. 45



Then, by similar reasoning to that previously employed, it may be proved that:

Area of element section

$$= p = (y_1 y_2 - y_0^2) \tan \beta$$

and $$ep = y_1 y_2 (y_1 - y_2) \frac{\tan^2 \beta}{3}$$

$$\therefore V = \tan \beta \int (y_1 y_2 - y_0^2)\, dx + \frac{\tan^2 \beta}{6\, r} \int y_1 y_2 (y_1 - y_2)\, dx.$$

By a simple transformation this expression is brought into such a form as to allow of mechanical integration. The final formula being:

$$V = \frac{\tan \beta}{2} U + \frac{\tan^2 \beta}{6\, r} W,$$

where

$$U = \tfrac{1}{2} \int (y_1^2 - y_0^2) dx + \tfrac{1}{2} \int (y^2_2 - y_0^2)\, dx - \tfrac{1}{2} \int (y_1 - y_2)^2 dx$$

$$V = \tfrac{1}{2} \int (y^3 - y_0^3)\, dx - \tfrac{1}{3} \int (y_2^3 - y_0^3)\, dx - \tfrac{1}{3} \int (y_1 - y_2)^3\, dx.$$

Another simple diagram has to be prepared, and by means of three operations of the integrator, the values of U and V are given thus:

$$U = 20 \times mn^2\ (v_1 + v_2 - v_3)$$

$$W = mn^3\ [320\ (u_1 - u_2 - u_3) - 100\ (w_1 - w_2 - w_3)]$$

where m and n have the significations formerly explained, and u, v, and w are the respective readings of the area, mo-

ment of area, and moment of inertia rollers in each of the three operations. Considering the great amount of calculation thus saved, and the accurate nature of the results, this second method, although involving rather more labor than the first, is a very important one.

The third method, which deals with the broken base, is much the same in principle as the second, but the expressions become more complicated, whilst six readings of the measuring rollers are involved. The case of an embankment, consisting partly of a cutting, is completely and accurately worked out by this method.

The Paper of Professor Amsler concludes with an example of the application of his integrator to the problem of the strength of a girder.

Continuous Integrators.

Any piece of mechanism which continuously adds up results may be regarded as a continuous integrator. Of such instruments, revolution counters as employed in meters of various kinds, form the simplest example, and correspond in action to the devices already described, by which the linear measurement of a boundary is performed. These will not be further referred to, and it is only necessary to consider those computing mechanisms which, dealing with the result of two simple unit measurements, correspond in principle to area planimeters.

It appears that Poncelet, before the year 1838, suggested the employment of a continuous integrator for computing the two factors in dynamometrical measurements. This was described by Morin in 1838, as applied in his "compteur" for registering the work done by a team of horses, dragging a loaded carriage at

any given velocity over any length of road. The principle employed was that of the disk and roller, the use of which as already shown, had been suggested for application in a planimeter more than twenty years before. In the case in question, the disk was turned by an endless cord or band from one of the wheels of the carriage, while the position of the roller on the disk was caused to vary with the tractive force, and its reading thus gave the product of force and space, or the actual work done.

In 1840, a Committee of the British Association, consisting of Professor Mosely, Mr. Enys, and Mr. Hodgkinson, was appointed to procure the dynamometrical apparatus of Mr. Poncelet, and to obtain a series of experiments on the duty of steam-engines by means of that apparatus, the sum of £100 being placed at their disposal for the purpose. The report of this committee, in 1841, describes at length the "constant indicator" of Professor Mosely, which was in reality a continuous steam-engine in-

tegrator. It was entirely a new instrument, except that the principle of the disk and roller was employed, and also the traction springs of General Morin. The pressure of steam was allowed to act upon a piston so as to vary the position of the measuring roller, while motion was given to the surface of revolution by means of the stroke of the engine. The surface of revolution was a cone, which was substituted for a disk, as by this arrangement the rapidity of the changes of velocity due to corresponding changes in the position of the integrating wheel is diminished in the same proportion in which the sine of one-half the angle of the cone is less than unity. The force necessary to drive the integrating wheel is diminished in the same proportion, and therefore the chance of an error arising from the slipping of the edge of the integrating wheel on the surface which gives it motion. The reports of the committee in 1842, 1843, and 1844 (which was joined in the first of these years by Dr. Pole), show that the action of the

above instrument was, as far as could be determined in its application to a single-acting Cornish engine, very satisfactory, but apparently no mention has been made of it, or results obtained from it, in any succeeding report.

Various other steam-engine integrators have been brought forward since then, most of them acting upon the same principles ; amongst these is the recent power-meter of Messrs. Ashton and Story, which is described at considerable length in the American edition of Weisbach's Mechanics as apparently something new. It is, however, the same instrument as Moseley's integrator, except that it employs a spiral instead of a straight spring, and returns to the use of the flat disk.

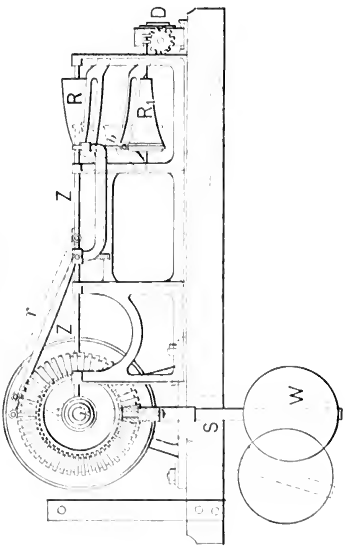
The disk and roller has also been since the time of Morin applied in many dynamometers, which thus become really "ergometers" or "work" measurers. In a series of articles which recently appeared in *La Lumière Electrique* those of Hirn, Megy, Bourry, and Darwin, are described as having a "totalizer" or in-

tegrator of this kind attached to them. The position of the measuring roller varies with the force exerted or transmitted, and the motion of the disk with the revolutions of the motor or machine. Thus by suitable counting apparatus the continuous product is given of force and space, or work done.

A cone has sometimes been used instead of the disk with dynamometers, as in that of Baldwin and Eickemeyer, used at the Centennial Exhibition in 1876 for testing mowing machines, in which, instead of a measuring roller, a cylinder is placed with its axis parallel to one side of the cone, while an endless band of round cord is rolled along between the two surfaces, so as to transmit the varying motion of circles of different radius on the cone. Fig. 46 shows the apparatus of Mr. Roury applied to a transmission dynamometer. In this the force is transmitted through a differential train of three bevel wheels, the middle one of which is attached to the spindle (S), which supports the weight (W), and is

suspended from the main shaft by the joint at (k), about which the whole can turn. Thus, the deviation of the weight

Fig. 16



from the vertical (as shown by the dotted lines) changes with the force. The change of position of the spindle (S) causes a band (b) to move along the sur-

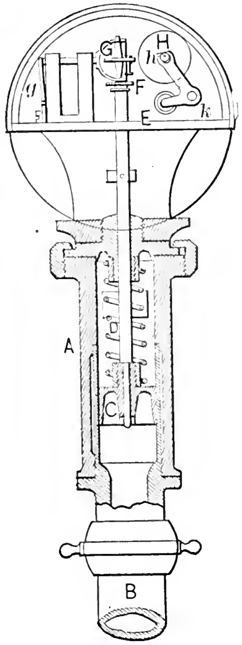
faces of revolution RR_1, the upper one, R, being turned from the shaft by the spindle (ZZ). It is to be noted that the distance of the band (b) from its zero position is not directly proportional to the force represented by the change of position of the weight, and, therefore, the surfaces must be formed with a certain curve, found by construction, in order that the dial and counting apparatus at D may correctly give the product of the two variables, force and space, and so the work transmitted through the dynamometer.

It cannot be said that continuous integrators of this kind are at present practically employed to any great extent. There are probably two reasons for this. One is the want of durable and reliable instruments. The other, the question as to how much, and to what degree they are really needed.

With regard to the first of these, it is evident that in all the arrangements hitherto considered (with the exception of Baldwin and Eickemeyer device) there

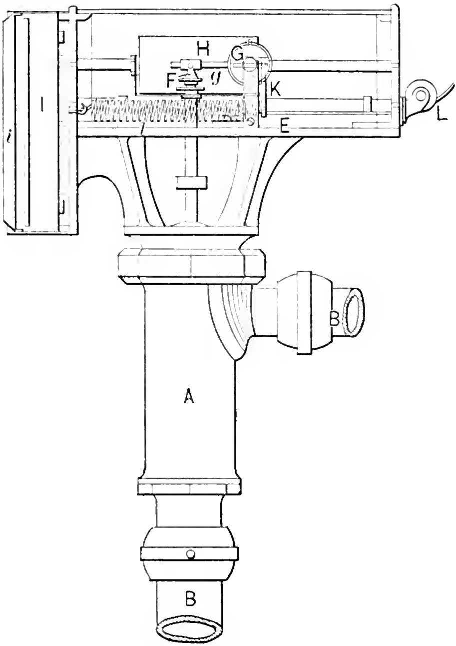
is that slipping of surfaces in contact, which, though of little effect as far as wear goes in the limited operations of a planimeter, becomes a very serious consideration when continuous action is required to be maintained. The only integrator of the second, or non-slipping class, which, as far as the author is aware, has yet been practically applied, is the "power-meter" of Mr. Vernon Boys. This instrument is shown in Figs. 47 and 48, and acts upon the same principle as Mr. Boys' integrator. The piston C, subject to the varying

Fig. 47



pressure in the engine-cylinders, with which the barrel A is connected by the connections at B and B', is moved up and down against or with the tension of the spring D; its rod acting on the arm *g* causes the plane of rotation of the roller G to take positions more or less inclined to the axis of the cylinder H. This cylinder H is moved to and fro with the stroke of the engine by means of the cord L, Fig. 48, and the roller G being in frictional contact with it causes it to turn round to a greater or less extent, according as the plane of G is more or less inclined to the axis of H. The amount of its revolution is registered by the counting apparatus in I (Fig. 48), to which the axis of H is geared, and is thus a measure of the power of the engine, for it gives the product of the tangent of the angle to which G is inclined and the distance moved through by H, that is the product of pressure of steam into the stroke of the engine. The steam being (as originally in Moseley's and also in subsequent integrators) supplied both

Fig. 48



above and below the small piston, the absolute pressure is given. Thus, in the present case, as the change of pressure on C at the beginning and end of each stroke causes the rod of g to be alternately above or below the axis of H, so the motion of the cylinder to and fro will always cause the cylinder H to turn in one direction, and thus to continuously integrate the work done. This device only enables a reciprocating movement of the cylinder H to be made, and the author has already mentioned the device of the sphere and rollers, which by the inversion of the higher pair of Mr. Boys, enables continuous motion to be obtained, and is suitable for application in dynamometers, electric-motors, and other purposes.

With regard to the want of such instruments, a very strong case was made out by the committee, already mentioned, in their report in 1841, where the application of a continuous integrator to steam-engines was alone discussed. The application has been made to electric-motors,

and in trials of motors and machines generally, and there is little doubt if continuous integrators combining the three qualities of durability, accuracy and cheapness could be produced, that in these days of increased regard for measurement of all kinds, there would be a much larger and increasing application of them.

Limits of Accuracy of Integrators.

In all calculating machines, accuracy of the result must be the question of first importance. Assuming the theory relied on in the various instruments for the mathematical operation to be correct, the accuracy depends primarily upon the mechanical arrangements, though in the case of planimeters it also depends upon the skill and care of the manipulator, and involves the question of a personal error. This latter point need not be considered, partly because this occurs more or less in all results obtained by observers, but also because it is less than might be

at first anticipated, from the fact that in tracing the pointer around the curve there is no reason why the error due to moving it on one side should exceed that due to moving it on the other side, that is, why equal errors of opposite effect upon the final reading should not be made.

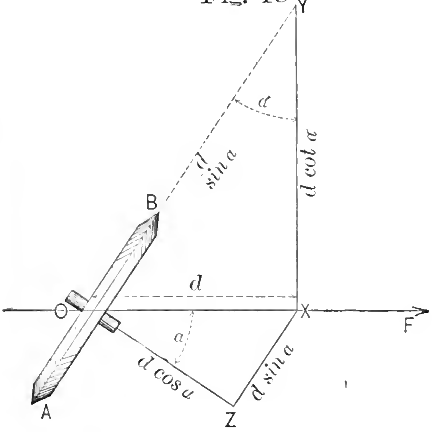
It has been seen that the action of all integrators, except mere revolution counters, depends upon the motion of the measuring roller, or its equivalent, over surfaces of various forms, therefore the above-mentioned mechanical arrangements resolve themselves into an examination of the nature of the frictional contact of two surfaces. It was for this reason that integrators have been classified according to the nature of this frictional contact, and it now remains to investigate the nature of this, to show to what the classification leads, to give the direct results of experiments upon the subject, and also the indirect results obtained from the instruments themselves.

Planimeters and integrators generally have been divided into—

I. Those in which the frictional surfaces slip as well as roll over each other.

II. Those in which slipping of the

Fig. 49



surfaces is supposed not to take place.

The order of this arrangement was adopted upon historical grounds, and also because the former class is at present by far the most important; but it would be more convenient, upon mere

grounds of mechanical simplicity, to invert the order.

Let AB (Fig. 49) be the plan of the measuring roller. Suppose a force applied in the direction OX, making an angle (a) with the plan of the axis of AB.

Let d=OX=distance through which the force acts.

1st. Suppose that frame which carries the measuring roller is free to move in any direction horizontally, but maintains the plane of rotation of the roller vertical, then the application of a force along OX, at the center of AB, will cause it to roll along the line coinciding in direction with the plan of the center line AB of the roller, that is, along the line OY. This will always be the case, except when this force is applied in the limiting case in the direction perpendicular to the plane of AB (*i. e.*, when a=O).

Thus, the distance in this case traveled by the center of AB, which is the same as the path rolled by it, is

$$OY = \frac{d}{\sin a}.$$

and the distance moved through by the center at right angles to OX is

$$XY = d \cot \alpha.$$

The latter value is the one usually taken or recorded by the instruments at present in use, but depends directly upon the former.

Next, suppose the frame carrying the roller is constrained either by guides, as in the linear planimeter, or by the radius bar of the polar planimeter, or otherwise, to move in the direction of OX, that is, in the direction in which the force acts. When the center of the roller has reached the point X, that is, when the force has been exerted through a distance OX,

Then OZ = distance slipped by AB = $d \cos \alpha$.

XZ = distance slipped by OB = $d \sin \alpha$.

Upon the degree of accuracy with which the above conditions are fulfilled depends the correctness of the working of all integrators; for not only do these two cases entirely cover the action of

the two classes of planimeters, and the corresponding continuous integrators, but one of the limiting cases in each, viz., that in which the force acts in the plane of rotation of the wheel (when $a=90°$), represents the conditions under which the wheel of the boundary measurer or opisometer is employed. It may be therefore said that the theory of mechanical action of integrators is based upon one or other of the following assumptions, in which the limiting case (namely, when $a=90$), is included.

Class I.—That the rolling of the planimiter, when slipping is allowed, is

$$N_1 = k_1 d \sin a.$$

Class II.—That no slipping takes place, which amounts to the assertion that

$$N_2 = k_2 \frac{d}{\sin a},$$

N_1 and N_2 being the readings in each case, and k_1 and k_2 suitable constants for the instruments.

It is easy to see that the first of these

is really the assumption made for all instruments in Class I.; but in the various instruments in Class II., it is only with the planimeter of Mr. Boys that it becomes directly obvious that the above assumption is made. With the others, though it is less evident, nevertheless, it will be found, on examination, to be equally true that the second supposition is really made, and that upon its truth the correct action of all instruments in the second class depends. The forces acting in each of the two cases must therefore be taken into consideration and the mechanics of the problem examined.

Proceeding in order of simplicity, Class II. will be examined first.

Let AB in both cases (Fig. 50) be the plan of the measuring roller.

Let S = reaction of surface upon which AB rolls, that is, the force with which it is kept in contact with it;

μ = coefficient of friction between roller and surface;

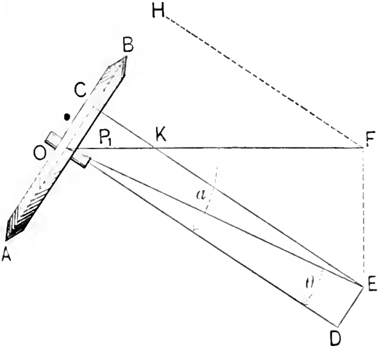
P = OC = reaction of surface, which must be brought into action in a horizontal direction to cause the roller to turn on its axis.

Class II. (*Fig.* 50).—Suppose the frame in which the roller is carried to be free to move in any direction horizontally, let a force be gradually applied at the center of the roller AB in the direction perpendicular to the plane of rotation. This will produce no effect as long as it is less than the maximum resistance, which can be opposed by friction between the edge of the roller and the surface upon which it rests, that is, as long as

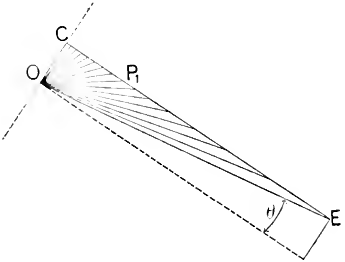
$$R = OD \text{ (Fig. 50) } < S\mu.$$

When the force R is equal to $S\mu$, and acts within the angle θ, the roller AB will move with uniform motion along the line of action of the force without turning. The same thing will hold if, instead of the force acting perpendicularly to the plane of rotation, it acts at some oblique angle to it not exceeding a cer-

Fig. 50



CLASS 2.



tain value measured from the normal. The limiting value of this angle depends on the resistance of the friction of the axle of the roller AB to turning. Let this angle be (θ), and draw OE perpendicular to OC, meeting the circle drawn with O as center and radius OD in E, and let

$$\theta = \text{angle EOD}.$$

When the line of action of the force falls without the angle θ, as, for instance, when it takes place in direction OF, the roller will still slip along the line of the force, but the roller will now also turn. The component in the plane of rotation will now, however, be of a magnitude such that the motion of rotation of the roller is no longer uniform. Since only uniform motion is being considered to take place, the conclusion is, that when the force acts at an angle greater than θ to the axis of rotation (*i. e.*, when $\alpha > \theta$) it must never be so great as to cause the roller AB to slip, and therefore only a motion of pure rolling can take place. By proper mechanical devices the roller can be made to turn very easily, and angle be kept very small.

The magnitude of the force which must be applied in any position of the roller to effect this motion, is

$$P_1 = OK = \frac{OC}{\sin \alpha}$$
$$= F \operatorname{cosec} \alpha,$$

and is at once given by the intercepts drawn from O to CE in the construction, shown in Fig. 50, for any other value of α.

Class I. (Fig. 51)—Suppose that the frame does oppose restraint, and that this restraint is such as to always cause the center of AB to move in the direction in which the force acts. Let OF (Fig. 51) lie in this direction, making the angle α with the axis of AB, draw EF perpendicular to OF from the point E, then by the triangle of forces. The force required to move AB is

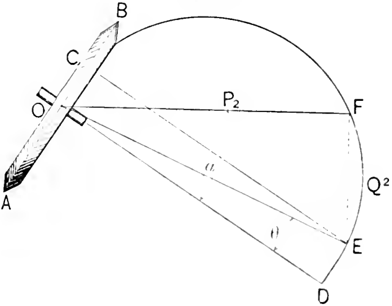
$$P_2 = OF = S\mu \cos(\alpha - \theta).$$

The reaction which is supplied by the frame is

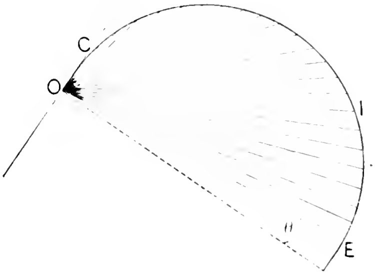
$$Q = EF = S\mu \sin(\alpha - \theta).$$

By describing a semicircle upon OE a

Fig. 51



CLASS 1.



construction is at once given, as shown in Fig. 51, in which, by drawing radial lines from O, the value of P_2 for any angle is at once given by the intercept.

The above investigation is by no means a complete one, for this would require a discussion of the moments acting, but having obtained the above result by the more complex method, the author considered it unnecessary to introduce a more detailed proof than has been given.

Comparing the two foregoing cases by means of the diagrams in Figs. 50 and 51, Diagram AB, it is clear that the forces acting always differ, except in the limiting cases; then

If $\quad a=90° \qquad P_1=P_2=F=OC;$

$\quad a=0° \qquad P_1 \text{ is } <S\mu;$

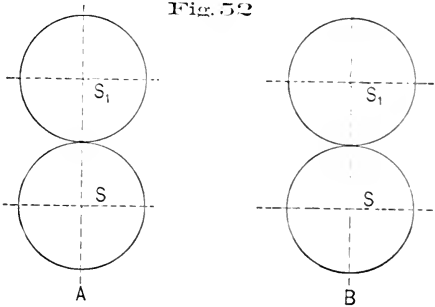
$\qquad\qquad P_2 \text{ is } \geqq S\mu.$

In order to examine to what extent the assumptions hold good upon which the accuracy of integrators rests, the experimental determination of the following points is needed:

I. Case of boundary measures and limiting case of both classes of area and moment integrators.—The conditions of rolling of two surfaces when the force acts in direction of the plane of rotation.

II. Class I. of area and moment plan-

Fig. 52



imeters.—Whether the rolling is proportional to the slipping, as assumed.

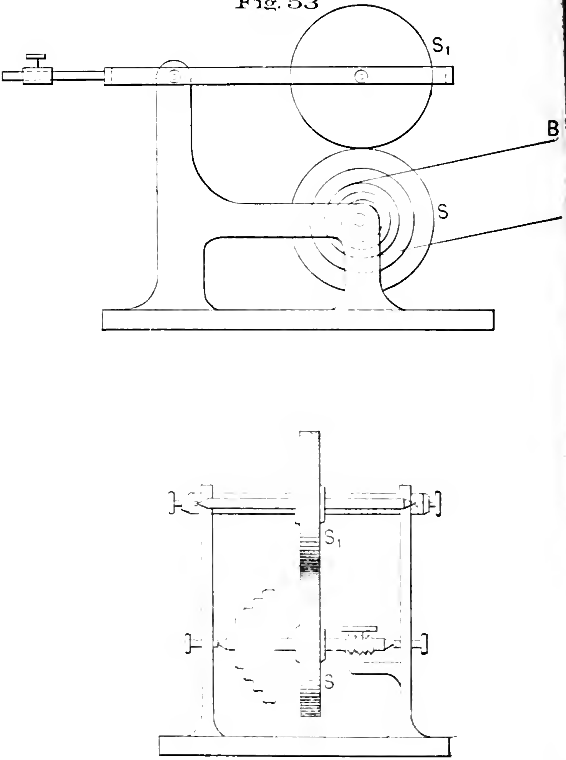
III. Class II. of area and moment planimeters.—Whether there is any slipping in this case, and if so, to what extent.

Case I. There are two separate problems under this head; one, where the planes of revolution of the two rolling

surfaces coincide; and the other, when they do not coincide.

Dr. A. Amsler has experimentally investigated the former thus: Two accurately turned disks, SS', (Fig. 52), each 200 millimeters in diameter, were placed with their edges in frictional contact, so that the mark shown on each coincided at the point of contact. By means of the apparatus (Fig. 53) the lower one S was turned by means of a band (BB') through 1700 revolutions one way (thus turning the upper one S_1), and then 1700 revolutions in the opposite direction. The marks which should have coincided if no slipping had taken place, were now found to be as in Fig. 52, the marks being a distance $= \varepsilon_n = 0.05$ millimeter apart. The relative error is thus $= \dfrac{\varepsilon_n}{2n\pi 200} =$ (about) $\dfrac{1}{40{,}000{,}000}$. This experiment is only of value to show the error when the wheel travels back by turning in the opposite direction, and at the most, shows that the error is nearly the same in both

Fig. 53

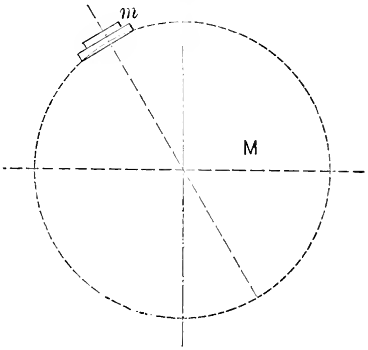


directions, and does not prove anything with regard to the action of the measuring roller of the boundary measurer; concerning this point, observations were wanted upon the results of the first 1700 revolutions. A very good way of examining the point would be to note the divergence of the marks in x revolutions when S drives S_1, then cause S_1 to drive S back through the same number of revolutions, and it would be seen whether the divergence was due to a difference in the periphery of the wheels, or to a slipping of the surface of contact. During Dr. Amsler's experiments no one was allowed to enter the room, in order to avoid alteration of temperature. He found that the heat radiated from the human body, or even from a lighted candle placed at some distance, had a perceptible influence on the result.

Fig. 54 shows a most important case, in which the plane of rotation of the surfaces in contact do not coincide with that of the measuring roller (m) being actuated by the disk M. Fig. 55 shows Dr.

Amsler's apparatus for examining this case, in which, if the edge of the roller has any width, there must be slipping action, even though the force always acts

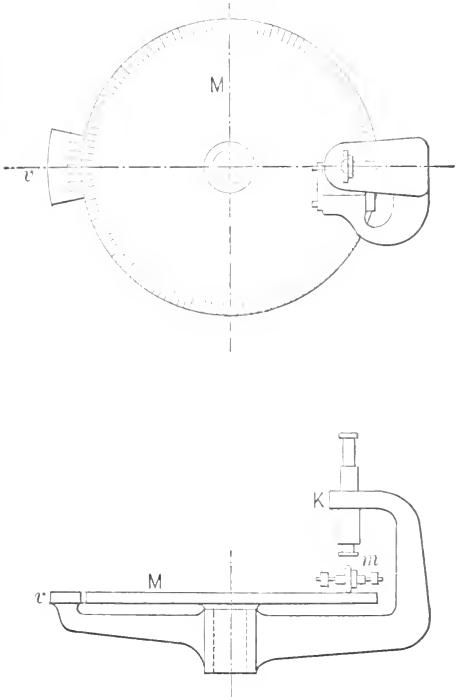
Fig.54



in the direction of the plane of rotation.

The roller m which rests on the upper surface of the disk, which latter has its edge divided, and is in juxtaposition with a vernier (v). The axis of the roller is fixed, and its edge is thus kept always vertically under a microscope (K). The

Fig. 55



position of the disk is noted, and it is then moved forward about 8 revolutions (or exactly 2,900°), which gives the roller about 130 revolutions, and a mark is observed on the latter. Then in theory the result of giving 8 more revolutions to the disk in the same direction should be to bring the same mark of the roller under the microscope, Practically the successive motions of the disk will be a little different, so that the second advance of the disk will not be exactly the same as in the first case. The same mark on the roller is, however, always brought under the microscope, and the difference in turning of the disk is what is noted.

In the following table—

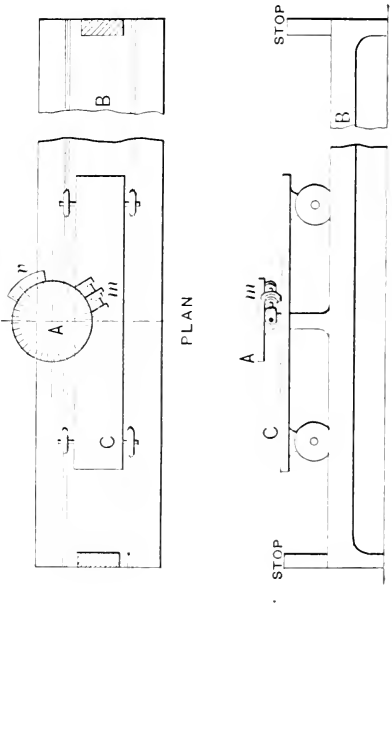
i=number of experiment,
φ=angle by which the disk differs from last reading,

so that the second column gives the positions of the disk at the end of successive advances in which the roller is made to take 130 complete revolutions, the third column shows the travel of disk in minutes (2,900° having, of course, to be added to the readings). The fourth

gives the difference between these and a mean value. The last gives the ratio of these differences to the travel.

$i.$	$\phi_i - i\,2{,}900^\circ.$	$\phi_i - \phi_{(i-1)} - 2{,}900^\circ.$	$m - (\phi - \phi_{i-1}$	$\dfrac{m - (\phi_i - \phi_{(i-1)}.)}{2{,}900^\circ.}$
	′	′	′	
0	2.58	47.22	−0.06	−0.0000003
1	49.80	47.28	−0.12	−0.0003007
2	37.08	47.34	−0.18	−0.0000010
3	24.42	46.98	+0.18	−0.0000010
4	11.40	47.22	−0.06	−0.0000003
5	58.62	47.04	+0.12	+0.0000007
6	45.66	46.98	+0.18	−0.0000010
7	32.64	47.34	−0.18	+0.0000010
8	19.98	46.86	+0.30	+0.0000017
9	06.84	47.34	−0.18	−0.0000010
10	54.18			

Fig. 56



Case II. To test the results when a roller partly rolls and partly slips, Dr. Amsler used the apparatus shown (Fig. 56) in plan and elevation. In this C is a carriage, running upon four wheels, on the base (B), which has parallel grooves planed in it; the travel of the carriage being limited by two stops at the ends. Upon the surface of C the measuring roller (m) rests, being attached to a plate A. By means of the graduations on A the axis of (m) can be set at any required angle with reference to the direction of motion of the carriage. The frame supporting the roller is carried on the disk by means of pivots, so as to allow (m) to rest on the surface of C with the constant pressure of its weight.

If a=angle of axis of (m) in the direction of the motion of the carriage;

s=motion of the carriage;
u=turning of the roller;

Then $u=s \sin a=s \cos\left(\frac{\pi}{2}-a\right)=s \cos \beta.$

If φ=actual reading of vernier v;
and $\varphi=\varphi_0$ when $a=90$ or $\beta=0$;
then $u=s\cos(\varphi-\varphi_0)$,

$u_1=s\cos(\varphi_1-\varphi_0)$,

$u_2=s\cos(\varphi_2-\varphi_0)$, etc.;

Then $\tan\varphi_0=\dfrac{u_1\cos\varphi_2-u_2\cos\varphi_1}{u_2\sin\varphi_2-u_1\sin\varphi_1}$,

$$s=\frac{u_1}{\cos(\varphi_1-\varphi_0)}=\frac{u_2}{\cos(\varphi_2-\varphi)_0}$$

In the following Table, which represents the results of experiments when the disk was covered with a surface of pear-tree wood. carefully polished (paper being, however, found to afford almost as good results):

i=as before, the number of the experiment;

B_i=angle of inclination of roller for experiment (i);

u'_i=motion of roller as observed for experiment (i);

u_i=motion of roller as calculated.

i.	$\beta_i = \varphi_i - \varphi_0$. °	′	″	Experiment u'.	Calculated u_i.	$u_i - u'_i$.
0	0	0	0	9,649	9,649	0
1	8	9	36	9,552	9,551	−1
2	9	36	0	9,511	9,514	+3
3	26	9	36	8,657	8,661	+4
4	27	36	0	8,550	8,551	+1
5	44	9	36	6,916	6,922	+6
6	45	36	0	6,750	6,751	+1
7	62	9	36	4,505	4,506	+1
8	63	36	0	4,290	4,290	0
9	80	9	3	1,650	1,649	−1
10	81	36	0	1,415	1,410	−5

Class III.—The actual conditions of motion when a force smaller than the component of $S\mu$, acts obliquely to the plane of rotation of the measuring roller, do not appear to have been made the subject of direct experiment. It is ap-

parently always tacitly assumed that no slipping takes place. But this crucial point cannot be thus left to mere conjecture, and the author has designed a method of carefully testing this, which he has not yet been able to properly carry out. From a few rough observations, there seems little doubt, howevei, that some slipping always does take place, and that its amount is, in the limiting cases, by no means inconsiderable.

Lastly, a few words may be said concerning the work of Professor W. Tinter and of Professor Lorber. The former has examined most carefully no less than nine different planimeters, from which he concludes that the different angles at which the measuring roller of the polar planimeters has little effect upon the result, and that, taking one turn of the measuring roller as $f=100$ square cm., the average error in the reading was only from $=0.00075$ to 0.0013, according as the center of rotation was without or within the area to be measured. The

work of Professor Lorber is so extensive and elaborate that it is impossible to do more than give in the most brief form the results at which he has arrived after many thousands of experiments. He concludes that error in the reading is always represented by an equation of the form—

$$dn = K \cdot + \mu\sqrt{n},$$

dn = the error in the reading, K and μ being constants.

where n = the reading of the measuring roller;

the above equation gives rise to one of the following form:

$$dF_n = Kf + \mu\sqrt{Ff},$$

where F = actual area to be measured,

and dF_n = the error in the result expressed in terms of the area.

The following are the results given in his latest paper:

Linear planimeter $dF = 0.00081f + 0.00087\sqrt{Ff}$

Polar planimeter

$$=0.00126f+0.00022\sqrt{Ff}$$

Precision polar planimeter

$$=0.00069f+0.00018\sqrt{Ff}$$

Freely swinging planimeter

$$=0.00060f+0.00026\sqrt{Ff}$$

Simple plate planimeter

$$=0.00056f+0.00084\sqrt{Ff}$$

Rolling (Coradi) planimeter

$$=0.0009\ f+0.0006\ \sqrt{Ff}$$

The degree of accuracy represented by these results may be inferred from the fact that in one case of the last planimeter, when

$f=100$ the relative error$=\dfrac{dF}{F}=\dfrac{1}{13{,}330}$.

Discussion.

Sir Frederick Bramwell, President, said that the paper having been read only in abstract, there had been no mention whatever of what the author himself had done. The members would no doubt, under the circumstances, allow him considerable latitude in personally explaining the apparatus on the table. He would not, however, ask them to defer the expression of their thanks to Professor Shaw for his valuable paper until this explanation was given; but he felt sure that after this was done, it would be still more clear that those thanks were well deserved.

Professor H. S. Hele Shaw said that the paper had been only read in the form of a brief abstract because from the nature of the subject, and its method of treatment, it appeared advisable that he should personally give a short account of its contents. The engravings, of which

there were a good many, could not be prepared in time to be sent to those who were likely to take part in the discussion, and, therefore, he would explain the principal points which he believed to be original, and which he hoped would be thoroughly discussed. He would take this opportunity of thanking Professor Amsler for kindly lending him several instruments, some of which were now shown for the first time in this country, having been sent from Switzerland for the purpose, at Professor Amsler's own expense. He also wished to thank Mr. C. V. Boys for lending him models of his tangent integrators, and also two of the actual instruments to exhibit; and his friend and colleague, Mr. C. D. Selman, for several valuable suggestions and assistance in the preparation of some of the diagrams. The author then proceeded to explain, by means of the diagrams on the walls, and by models which had been constructed on a large scale, the principles of the classification adopted in the paper and the various instruments exhibited.

Mr. William Anderson (of Erith) observed that he had had considerable experience with continuous integrators in measuring work done by agricultural implements. There was a good deal to be said about the use of those instruments and the defects to which they were liable, about the personal error, which was an important point, and errors from imperfect adjustment. The conclusion at which he had arrived with regard to continuous integrators, in which the space passed over and the strain were multiplied together and registered continuously, was that they were exceedingly good for comparative results, but were not altogether to be trusted for positive indications. In comparing, for example, a number of machines working in a field under similar circumstances as to weather and everything else, with the same operator, the comparative results would be trustworthy; but if there were any variation in any of the conditions, they would not. There was always a good deal of doubt about the positive results. The

causes of error were these. In the continuous integrators the integrating wheel was attached to a train of wheel-work which possessed a considerable amount of inertia and friction. In planimeters, and integrators of that class, where the observations could be made slowly, at a steady speed, and where the conditions did not vary, inertia did not count for much; but in the steam-engine, and in agricultural implements or in traction indicators, there were great and rapid variations of speed. The sudden strains which were put on by the tractive force shifted the integrating wheel along the disk suddenly, and the speed changed in a similar manner. The force necessary to accelerate the movement of the integrating wheel and its mechanism tended to cause a slip, which was partly counteracted by a slip produced by the force necessary to arrest its motion when the speed changed, but the friction of the mechanism always acted in the same direction, tending to augment the error. With one implement, for example, if there

were a tolerably steady pull with no sudden variation in the velocity, there would be one amount of error, whereas if another implement were worked in a jerky fashion there would be a totally different amount of error. With regard to personal error, to which the author did not attach much importance, he had reason to think that it was of great consequence. The degree of care and skill exhibited in adjusting the instrument and taking the measurements had an important influence ; he was therefore always careful, in a series of experiments with agricultural implements, to have the same observer throughout, if possible, because the results aimed at were rather comparative than positive. Still, he was bound to say that, with care and experience, satisfactory positive results could be attained. The author did not appear to be aware of the extensive use which the Royal Agricultural Society had made of the integrator which had always been known as Morin's, but it appeared that honor had been ascribed where it was

not strictly due; probably the reputation of the great French mechanic, who had done so much to introduce it, had obscured the claims of the real inventor. For the last thirty-five years the Royal Agricultural Society had used continuous integrators, and he thought that there was no one more competent to speak of their action than Sir Frederick Bramwell, who had himself conducted many of the experiments. The issues had often been very important, involving the fortunes of manufacturers of agricultural implements, which had, in a great measure, hung upon the indications of the dynamometers. Most of the apparatus used by the Royal Agricultural Society had been designed and constructed by the late Mr. C. E. Amos, M. Inst. C. E., and by his son Mr. J. C. Amos, who for many years filled the office of Consulting Engineers. With reference to continuous indicators for steam-engines, he had little or no experience. He had tried them, but the results had not been, so far, satisfactory. One of the chief defects

was the difficulty of keeping the little integrating wheel perfectly free from flats. It was not easy to find any metal perfectly uniform throughout; and if it were not uniform, a flat would soon form, and then all the results would be utterly untrustworthy, because the wheel tended to hesitate at the flat place. Formerly integrating wheels were made of gun-metal, cast with great care, and under a great deal of head. Latterly they had been made of steel, and a better result had been obtained. But even when the wheel wore uniformly, if the width of the surface in contact with the disk varied, then again there was a source of error, because the surface of the integrating wheel lying upon the disk became greater, and then there was uncertainty as to the true diameter of the periphery of the disk on which the integrating wheel was working. The only way of eliminating these errors was by repeated testing of the apparatus.

Mr. C. Vernon Boys said the first part of the paper on which he desired to say

a few words was the division of the subject into different classes. The author had referred to one system of classification which Mr. Boys had adopted in a paper published in the *Philosophical Magazine*, a division into three classes. The author had rightly shown that the two classes which Mr. Boys had called the "radius class," and the "Amsler class." were in a mechanical sense, that was, so far as the connection between the surface of the integrating wheel or roller and the surface on which it worked was concerned, absolutely identical. But though that was undoubtedly the case, he thought the division might still hold good, for an inventor could not have contrived machines in one class or in the other without having had some such system of division in his mind at the time of the invention. There was one considerable omission in the paper, and that was the only point on which he felt it necessary to find fault. There was no mention of a very large series of most beautiful machines, designed, and he be-

lieved partly constructed, by the author himself. As those instruments had been fully described in a paper by Professor Shaw before the Royal Society, possibly he thought that they were so well known that it was unnecessary to describe them again; but it would certainly have rendered the paper far more complete and valuable if that large amount of work had been incorporated in it. Of all the instruments brought before them, he thought that the new precision planimeters and that extraordinary spider-looking instrument crawling on the sphere, were those which called for the utmost admiration. It was impossible to look at them or to use them without being impressed with the extreme mechanical beauty of their construction and design. But though that was undoubtedly the case, he thought that no one could see some of the combinations of a sphere with three, eight, or six little wheels round it, inventions of the author, without classing them even above those instruments in point of beauty.

The author had spoken of the tangent principle as being a particular case of machines which worked without slipping. This was perfectly true; but there was a very great distinction between integrators depending upon the tangent action and integrators depending upon a rolling action of the radius class, to which alone the other machines belonged. There was one accidental mistake in the wall diagrams representing two spheres which he desired to point out, as it was a little puzzling. In the instrument of Clerk Maxwell, one of the non-slipping radius class, the axes of the two spheres were shown in such a position that rolling would ultimately make them coincide. In reality, the equator of one rolled over the pole of the other, as was obvious to those who had anything to do with integrators. The distinction between instruments of the tangent class and those of the radius class might be represented by the little wheel of the instrument. The accuracy of integrators of the radius class depended upon the exact size of the wheel, and the

exact size of the surface upon which it rolled; and, as Mr. Anderson had remarked, it was very much impaired when flats formed upon any of the surfaces in contact, for there was then a little hesitation. The mere action of integrating machines in which there was slipping was sure to produce those flats at some time or other, so that the time they were likely to last and the amount of work they were capable of doing were limited. The actual size of the roller was of importance, for as it wore and became gradually smaller, the number of rotations were affected, and therefore the recorded result gradually increased. These were the objections to instruments of what he had called the radius type. In the class of instruments at which he had worked exclusively, and which he believed he had originated until he found that to a certain extent Mr. B. Abdank-Abakanowicz had preceded him, the wheel was allowed to roll along just in the direction in which it was pointing. Anyone who had been in the streets of London must have noticed

with what extraordinary persistence and power the wheels of all vehicles went in the direction in which they were pointing. A butcher's cart driven round a corner with tremendous fury would often, in spite of its jumping on the ground, still continue its course with very little side slip. If the springs were sufficiently good, or the road sufficiently smooth, so that the wheel kept in contact with the ground it would apparently not slip at all. In the case of the first instrument which he had made, which he had called the cart integrator, he was astonished to find with what extraordinary accuracy results were obtained by its use, such, for example, as the area of a circle and other things that were known. In that instrument he had depended entirely upon the fact that the steering wheel of a tricycle would go along in the direction in which it pointed. The instrument had to pull a heavy brass cart after it and slide in a comparatively roughly-made groove, but even so he found the value of π, on squaring the circle, came out 3.14 when using

a very rough home-made instrument. It seemed, then, that if the sources of side slip which were undoubtedly present, due to the great friction that had to be overcome in dragging the cart, could be removed—that was, if the ground under the cart could be made easily movable, which was the case when it was converted into a cylinder, the cylinder would follow exactly in the direction in which it should go. For that reason an integrator of that class was free from the objections which had been raised, that in time integrators wore out, the little wheel got flats upon it, and as the size varied the record varied. In the case in point there was no slipping and no tendency to make flats, and if the wheel was made half or twice the size the record was the same, for it only depended on the direction in which it wanted to go, not upon the amount that the wheel turned in going along in that direction. He desired to say a word or two with regard to an instrument, his engine-power meter, which was not so well known as he had hoped

it might be. There were certain sources of error apparently present in it, but which were really imaginary. In the first place, it would seem that if the position of the piston-rod, and, therefore, the angular position of the little tangent wheel G varied in the least, then as the cylinder H traveled along there would be an error due to the angular want of true precision. But that was not so, for supposing the spring to be too long or too short, and the tangent wheel to be permanently deflected, when there was no steam pressure, at an angle, say, of 2° or 3°, then as the cylinder moved in one direction, the wheel would run up a certain slope, and when the cylinder moved in the other direction, it would run back along the same slope, and so far as that was concerned, no error would be introduced. The wheel might be set at a permanent angle, and the roller work backwards and forwards, and nothing would be recorded at the end of the operation. He could show it with a rough paper and wood model. If he permanently compelled

the tangent wheel to assume a certain angle by holding it with his thumb, as the cylinder traveled in one direction, it would rotate a certain amount, and when it traveled in the other direction it would rotate back again, and on the dial there would be no permanent record. The disk-cylinder integrator had one serious defect; it was very difficult to apply it to cases in which growth of time or of motion was continuous. If it was desired to have a time-integral and a motion-integral, when the motion was continuous, it could not be easily applied, because when the cylinder had got to one end of the stroke, there was nothing for it but to stop and come back again. It was, therefore, necessary to apply a mangle motion, so that as the motion was continuous the cylinder went backwards and forwards; then by causing the cylinder to work between two tangent wheels alternately, or by letting the mangle-motion work ordinary reversing gear between the cylinder and the recording mechanism, continuous integration could be ef-

fected. But of course the instrument, where continuous integration was concerned, would not compare with the spherical integrators designed by the author. On the other hand he did not think that any instrument, in the peculiar case presented by the automatic integration of an engine diagram, could compare with it, for those peculiarities of motion which interfered with the ordinary form of radius machines, by causing a perpetual scrubbing, to which reference had been made, produced no trouble at all, for there was no side-slipping, and because the very large moment of the radius integrator was replaced by the extremely small moment of a little cup containing a wheel not the size of a three-penny bit, and by a little bar of steel. In fact, the moment of the piston and its attachments was nothing like so great as that of an ordinary Richards Indicator, because there was no multiplication of motion.

Mr. W. Anderson wished to state that he had been able to bring to the meeting the integrating part of one of the old

dynamometers of the Royal Agricultural Society. The date of its construction was unknown, but he believed it was about the year 1848.

Mr. J. G. Mair observed that the author had not given himself sufficient credit for the machine he had invented, and he would like to ask him if the reading of the counter gave such accurate results as the vernier on Amsler's planimeter. To read by means of a counter was of great advantage, and especially so where large numbers of indicator diagrams had to be taken out; on some of the engine trials he had made, three hundred diagrams had to be averaged, and constantly reading the vernier on Professor Amsler's instrument was trying to the eyes. The application of the integrating machine as a power meter was a most useful one, and he had made several trials with the one invented by Mr. C. Vernon Boys; on three trials the readings were 68.5, 68.2, and 73 by the meter, against 67, 67.5, and 72 in the pump. The pump-power was taken from the dis-

placement of the pump piston, and the head as shown on a mercury gauge. He thought those results were a proof of the correctness of the instrument. He did not think that reading a counter was more difficult than reading the dial of a watch. One other measurement had to be made, namely, the distance between the stops, and as that could be measured with an ordinary rule, a child could almost make the simple calculation, so that he did not think personal error very much affected such an instrument. There was naturally a good deal of diffidence in using a machine, the details of which were not thoroughly grasped. With an indicator diagram, there was something to see which was readily understood; but where compound measurement was shown on a counter, it was at first difficult to realize the reading as correct. As soon, however, as the instruments were better known, he had no hesitation in saying that, where the absolute shape of an indicator diagram was of no consequence, the continuous integrator would entirely

supersede all other means of measuring power.

Mr. Druitt Halpin said that the author had referred to one of the minor improvements by putting a locking-spring to the frame. Mr. Mair had noticed the great difficulty there was in reading the instrument, on account of the small scale to which it was graduated, but the author had not completely followed the idea of adding the locking arrangement. The convenience of reading the instrument was doubtless very great, for it could be taken up and put in a good light, but the real object of the locking gear was to make the instrument do its own addition, and also carry over the decimal places now lost. Instead of taking each diagram by itself, and writing down the result of each separately, and adding them up, the instrument was set at 0 and and locked; the instrument was run over the diagram, and it was locked, and so on. So that by taking ten diagrams successively, and putting the decimal point one point back, the instrument was made to do

its own addition It had the further advantage that, whatever the last decimal might be, it was carried on by the instrument. If it was 2.38, it could not be said whether it was 2.389 or 2.381. With regard to Mr. Boys' application of these instruments, he had had an opportunity of testing it on an engine of 120 or 130 I. H. P., and he found the results coincided within from 1 to 2 per cent. of the indicated power which was obtained with standard indicators. With Mr. Webb's permission, he also put one of his powermeters on the large rail mill at Crewe, and took a series of observations there of the exact power required to roll a rail, from the moment the ingot touched the rolls to the moment the next ingot touched them, which gave the true power, correction being made for any difference of velocity in the fly-wheel. It gave the power both while the engine was running with the bloom in the mill and when it was running empty. Whether the machine in its present form would be suitable for locomotives, he was hardly

prepared to say, because he feared that the attachment which was provided for taking diagrams was, perhaps, so heavy that its inertia would interfere with the correct action of the instrument. But, if that was lightened, or the attachment left out, he was sure the best results would be obtained from it.

Mr. H. Cunynghame said that his attention had been called a great deal to Mr. Boys' machines, in the development of which he had assisted, as they were first designed for practical application to steam-engines. He had made many attempts to apply continuous integrators for the purpose of integrating electric power, and he certainly could safely bear out all that Mr. Anderson had said about the imperfections of integrators of the slipping type. If a wheel was running in any direction in which it was likely to slip upon the surface, it would be rubbed into facets, and in like manner, a sphere would be rubbed into a polyhedron. In this state both wheel and sphere would be worse than useless for integration.

But if these machines had an integrator of the roller type, even if it had facets already, the rolling would take the facets out of it just as an apothecary rolled his pills. Instead of an integrating wheel of hard steel, as was necessary in all integrating machines of the slipping type, in those of the rolling type, soft metal might be used, and the more it was used the better it would get. Moreover, in such machines, since what was counted was not the revolutions of the wheel, but the revolutions of something caused to roll by means of the wheel, the accuracy of shape of the rolling wheel became of minor importance, and even if it "skated" over the surface instead of rolling, a fair result would be obtained. He thought Mr. Anderson's remarks were somewhat unjust to planimeters of the rolling type, which he did not think had been brought before the public so as to be within the experience of anyone. That led him to say a few words on Mr. Boys' steam-power meter. He believed that Mr. Boys and he were the first in this country to

make a trial of those machines. They made a trial at the works of Messrs. Ransome & Jocelyn at Battersea, and for that purpose they had an extremely good indicator made by Messrs. Elliot. It would be quite understood that Mr. Boys' indicator was not in any way calibrated by mere trial. They took the measurements of the engine and the diameter of the cylinder. Then they took the amount by which a certain pressure of steam would cause the spring to rise, just as was done before taking the H. P., by means of a Richards Indicator. They thought that if those two machines were used upon the same engine, and if the results given by the two machines, namely, Richards and Boys, were independently calculated from the constants of the two machines, then, if these results nearly corresponded, a very remarkable coincidence of testimony would be obtained. They therefore put Boys' machine on the engine alternately with Richards', and he thought they must have alternated twenty times, and the results of Boys' machine

were found to be uniformly, during the first series of experiments, about 23 per cent. too high. That puzzled them extremely, but upon examination they found that the manufacturer, instead of making the piston 1 inch in diameter, had made it 1 square inch in area, and when they made the proper correction they found the results to agree to about $1\frac{1}{2}$ per cent., and with those stated by Mr. Halpin and Mr. Mair. Mr. Mair's tests were made in this way: he raised, by means of a pump, a given weight of water through a number of feet, and then he estimated the foot pounds and compared them with one another, and the result showed a remarkable degree of accuracy, the discrepancy of one or two per cent. being accounted for by the loss of work owing to the raising of the pump valves, which were large and heavy.

Dr. William Pole, after testifying to the high character of the paper, offered a few remarks on the very early example of mechanical integration with which the author had connected his name. It had

come about in the following manner: Some half century ago the engineers of the center and north of England became aware of the reports published from time to time of the extraordinary economy of the pumping engines of the mines of Cornwall. These reports at first obtained no credence, and even when they were found to have some foundation, the most singular attempts were made to explain them away. In the midst of the controversy, the late Mr. Thomas Wicksteed, M. Inst. C. E., the Engineer to the East London Water Works Company, determined to throw light on the question by buying an engine in Cornwall, and setting it up to pump water on his own premises at Old Ford, where it could be thoroughly tested and examined.

The subject had previously been brought before the notice of the British Association for the Advancement of Science, and had attracted the attention of Professor Henry Moseley. He, in writing his excellent work on "The Me-

chanical Principles of Engineering and Architecture," had become acquainted with a principle of dynamometrical admeasurement proposed by Mr. Poncelot, and carried out in 1883 by General Morin; and it occurred to him that a machine might be contrived on a similar principle, applied to record the work done by a steam-engine, by a species of mechanical integration, combining the pressure exerted on the piston with the space moved through; and it was seen that such a machine would be most usefully applied in testing the performance of Mr. Wicksteed's Cornish engine.

At the meeting of the British Association in 1840, a grant was made for the purpose, and a committee, consisting of Professor Moseley, Mr. Eaton Hodgkinson, and Mr. J. Enys, was appointed to carry it out. The machine was constructed under Professor Moseley's direction, and a full account of it was given in the report for the following year. The trials were then made on the engine, and the results were exceedingly satis-

factory. The integrator worked for a month without intermission, and its indications, when calculated out, were found to agree closely with the results obtained, as accurately as they could be, by other means.

The fixed data of the engine had all been well ascertained, but, to render the comparison complete, it was found desirable to get, if possible, an accurate measurement of the velocity of the piston at various parts of its stroke; this velocity was very variable, depending not only on the mass in motion, but also on the ever-varying force of steam acting on the piston, and on means which had hitherto been devised for ascertaining the velocity experimentally. The attention of the committee had, however, been directed to an admirable chronometrical instrument contrived by Messrs. Poncelot and Morin, and Professor Moseley undertook to adapt a machine on this principle to the Old Ford engine.

At this time Dr. Pole (who had been

occupied independently in investigating the action of the Cornish engine) had the honor of being invited to join the committee, and the two succeeding reports, in 1843 and 1844, were written by him.

By means of these two instruments, and by an ordinary indicator, aided by careful observations as to the consumption of coal and water, and other variable factors, the working of the Old Ford Cornish engine was investigated, both theoretically and practically, in a most complete and accurate manner, and the peculiarities of this form of engine, as compared with the ordinary engines in use at that time, were thoroughly brought out, so as to clear up all the doubts that had been so long entertained by engineers. This application of a mechanical integrator to real work of magnitude and importance was probably the earliest made. It was intended to follow it up by adapting the machine to other purposes, especially to ocean-going steamers, and it was actually fitted to

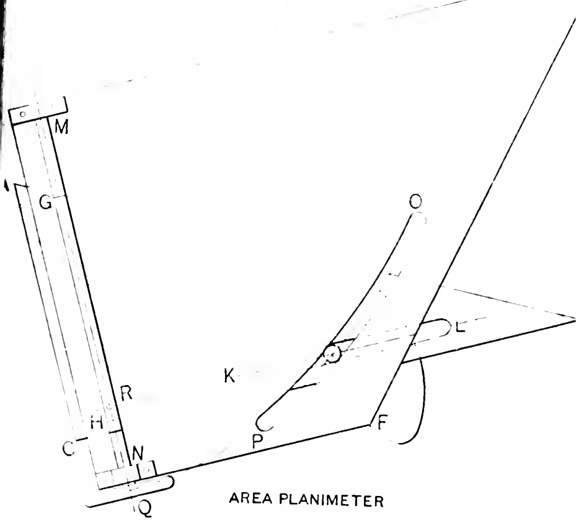
the engines of the Great Western steamship on her first voyage, but by an accident which it was difficult to repair at sea, the experiment was rendered useless, and the attempt was never renewed.

Mr. W. R. Bousfield said he had not had the advantage of reading the proof of Professor Shaw's paper, but he had heard it read in its short form at the previous meeting, and he thought possibly it might be of interest, and useful in connection with a paper which dealt, more or less, with the different types of mechanical integrators, if he mentioned a principle which, so far as he knew, had not been alluded to. It occurred to him some years ago that, by the use of various curves in connection with planimeters, various products and squares might be readily integrated. A simple form of area planimeter of this kind might be made by combining a T-square and a set-square, or two set-squares. Referring to the illustration, Fig. 57, the lower T-square or set-square ABC would

have in it a slot KL, and upon it a rib GH, and would be capable of motion about a point R in line with the slot. The upper set-square would have a slot MN capable of sliding upon the rib GR, and also a parabolic groove OP. A pointer—in practice a ring with cross wires—would slide in the slots KL and OP. In using this planimeter a needle point would serve to pin the lower set-square through the point R, at a convenient pole in relation to the area to be integrated. The cross-wires or pointer would be carried round the curve enclosing the area to be integrated, the two set-squares at the same time sliding on one another by means of the rib GH and slot MN. The measurement would be made by the roller Q placed at the vertex of the parabolic curve. The principle of the apparatus was very simple. Taking polar co-ordinates, with R as the pole, the area to be integrated could be expressed as

$$\int \tfrac{1}{2} r^2 \, d\theta.$$

Fig. 57



AREA PLANIMETER

If now the equation of the parabolic slot were

$$y^2 = a\,x,$$

and the parabola moved, as in the apparatus, so that y was always $= r$, the expression for the area would become

$$\int 2\,a\,x\,d\,\theta.$$

This showed that the area was measured by the roller Q at the vertex of the parabola. This proof rather suggested that possibly the use of polar co-ordinates in some of Professor Shaw's proofs would shorten them. By the use of other curves, instead of a parabola, the principle could be applied to the mechanical solution of various problems in integration.

Professor W. H. H. Hudson remarked that the classification adopted by the author coincided, if not entirely, very considerably with this, that in the instruments in which slipping was bound to take place, when it was attempted to express the element of area, in mathematical language $y\,dz$, dz corresponded to the

angle turned through and y to the radius, so that the area corresponded to the length. That was certainly so in some of the earlier instruments; but those in which no slipping was allowed to take place seemed to correspond rather to a different mode of translating the mathematical expression. There was a certain amount of turning which was proportionate to dz, and then by a suitable mechanism that was turned into something else proportional to $y\ dz$ by using a multiplier proportional to y. One wheel was made to turn so many times faster than another, the number of times corresponded to y instead of, as in the first class of instruments, the radius being proportional to y. That multiplier occurred in the various forms of the instrument, sometimes as a sine, and sometimes as a tangent. Those of the sine class would have this disadvantage, that the multiplier must, of necessity, be a proper fraction. That would limit the range, whereas those dependent on the tangent seemed to have no limitation

whatever, and the multiplier might be anything. This seemed to be a great advantage of the sphere instrument of Professor Shaw, that there was, apparently, no limitation at all, that which represented y being a multiplier which represented a tangent. He had always taken great interest in mechanical illustrations of mathematics. In most cases the mathematics had been incidental, but in this case it was of the very essence of the whole business. These were instruments designed to illustrate not only mathematical results, but even mathematical processes, and he was very much struck by the way in which the author had developed Amsler's planimeter out of the simple form of the disk and roller. It illustrated a mathematical conception which was sometimes received with incredulity. by those who met with it for the first time, namely, treating parallel straight lines as meeting in a point at infinity. This was done in pure mathematics, and he found that the author had been obliged to do it in order to bring

his instruments all under one head. It seemed to him that this was a striking illustration of the inter-action of mathematics and mechanics, and the benefit which each gave to the other.

Mr. W. E. Rich said the integration of the areas of plane surfaces, which these instruments were ordinarily designed to do, was much less difficult than the integration of work done on a dynamometer in the manner spoken of by Mr. Anderson. One of the great desiderata in dynamometrical measurements was to get a spring which followed Hooke's law; but, unfortunately, in practice they never got one that followed it precisely. The next thing was to adjust the instrument for zero. If the instrument were not accurately so adjusted, complex formulas had to be used for reducing the results, and as in many instances zero was not accurately determined, and these formulas were at the same time overlooked, the results recorded were in such cases very erroneous. One point of practical difficulty with in-

tegration for a dynamometer was the slipping of the small disk upon the large one. In the field, when it was raining, the observers were much hampered in that way, and on a stony road, when traveling at a high speed with a small integrating disk, the contact between the disks was frequently broken by the shake and jar of the whole instrument. These observations were perhaps not directly pertinent to the question of integration; but one of the most useful objects of integration was for dynamometrical purposes, and therefore he thought it worth while to mention them.

Professor H. S. Hele Shaw, in reply, said that the question of mechanical integrators was a very much wider one than many would at first think: but the discussion upon the paper had embraced two principal points: Planimeters such as were shown upon the table, and the subject of Continuous Integrators. The slipping class of planimeter had been brought to a great state of perfection, and their accuracy was, as he had shown,

remarkable. It seemed to him that the only objection to them—an objection which several speakers had adverted to —was the difficulty of reading them. Most of the remarks had been devoted to continuous integrators. Perhaps that was not altogether unnatural, because they had, as Mr. Rich had pointed out, such a directly practical application. Mr. Anderson had given valuable testimony on the subject of the defects of such instruments as were in use for dynamometrical purposes, and his experience agreed with that of Mr. Rich and of all who had much occasion to use them. Professor Shaw thought the general conclusion was that none of the continuous integrators in use for this purpose were really satisfactory in their action. To say of an instrument of measurement that it was not an accurate instrument, that it was merely capable of giving comparative results, was, after all, to give it very poor praise, for mere comparative results simply meant that no constant for the instrument could be obtained.

He had no doubt that, for temporary purposes, the difficulties might be overcome, but it was the general opinion that an instrument like the disk and roller, which, by the way, in spite of what Mr. Anderson had said, *was* first applied by Morin after its invention by Poncelot before 1840, was not reliable when in continuous use. An instrument that worked one day differently from another day, because the conditions were slightly changed, must certainly be pronounced to be very imperfect. It was because of this very difficulty that he was led, as he had no doubt Mr. Boys and others had been, to work at the subject, and to endeavor to overcome the difficulty of the slipping action. Mr. Mair, Mr. Druitt Halpin, and Mr. Cunynghame had all testified to the satisfactory action of the tangent integrator of Mr. Boys, which was one solution of the problem of avoiding the slipping action in question, but had been employed in a far more trying application than with a dynamometer, viz., a steam-engine integ-

rator. This he was very glad to hear, because the instruments of his own, exhibited on the table, depended upon the same principle for their successful action, and it was just in this particular that they differed from the integrator referred to by Dr. Pole. His instruments had been alluded to in complimentary terms by one or two of the speakers, and perhaps he might be allowed briefly to describe the principle upon which they worked, and also to call attention to two new instruments which were shown for the first time on the previous Thursday at the Royal Society. The difficulty which he endeavored to overcome was to make use of those mathematical properties of the sphere, which he had previously discovered and published, and which Professor Hudson had alluded to as to sine and tangent forms. As was not surprising, he had found, when he came to collect information on the subject, that the same properties had been employed by others, amongst whom were Professor Mitchelson and Professor Ams-

ler. But what the author had endeavored to do was to avoid the slipping of the measuring roller over the integrating sphere. That was his first thought. There was generally a germ in all inventions, and that was the germ of his. If, instead of moving the measuring rollers round the sphere, they could keep them in contact with the sphere, and move the axis of the sphere, or, as it at first occurred to him, have the two rollers in contact and roll them around the spheres, they would hold it in such a way that they would allow it to rotate around the different axes in the solid. At first he did not think it would work, but he tried it with a rough boxwood ball, and two disks with india-rubber rings around them, and he was surprised to see that the rollers did move round, and to see the ball rolling round on a different axis. He thought at first it was merely a question of rolling centers, but after some time he saw that it was really a question of geometrical principle, and having certain rollers in contact around one equa-

tor it did not matter where they were placed, so long as the others were perpendicular to the first, acting, in fact, on the same principle as that by which two beveled wheels would work together. The axis of rotation around which the sphere must turn, because of the contact of the other set of rollers, was in the vertical plane; there was but one intersection of the two plates, and that intersection was the axis of rotation of the sphere. That principle once understood enabled him to construct various other models based upon it. He was indebted almost entirely to his brother, Mr. Edward Shaw, Stud. Inst. C. E., and to Mr. W. E. Kerslake, for working out this in a practical form, and constructing the various instruments exhibited. One of these instruments was a continuous integrator, which had been designed for application to the dynamometer of the Rev. F. J. Smith, of Taunton, who intended to have been present and take part in the discussion, but from whom he had just received a telegram stating that

he had tried it most carefully and exhaustively, and had found that it had given absolutely uniform results. He had tried it himself, but it was a very satisfactory thing that it went to Mr. Smith straight from the hands of the maker, not adjusted in any way. It was remarkably simple. According as the arm was moved into a more or less inclined position, so there resulted a larger or smaller travel of the indicator. No appreciable effort was required to turn it; it was done with a minimum of power, and it entirely got over the difficulty of unequal wearing. He had every reason for thinking that, if anything, it would rather tend to wear the ball round than otherwise. Another instrument that he had on the table was the moment integrator. Professor Amsler's moment integrator was an expensive one, and at the time of writing the paper he had not any idea of constructing one: but di rectly he had made a small area planimeter, he saw that a much simpler instrument could be devised by employing the

wheels in a very small compass, and he should be happy to explain after the meeting, to anyone who might be interested, the performance of the operation by the application of these small wheels. Mr. Mair, he believed, asked whether the readings of the instrument were accurate. It had not been tried so exhaustively as Amsler's instruments, and therefore he could not compare it with them; but he saw no reason why they should not expect the same accuracy from it as from those instruments in the non-slipping class, concerning the accuracy of which they had already heard testimony. He greatly regretted that it had not been possible to prepare the engravings in time to be sent around with the paper, or some points he had been anxious to have discussed would, no doubt, have been dealt with more fully by the various speakers. For instance, the points which Professor Hudson had referred to were actually treated in the paper, and, he ventured to think, were, by means of diagrams, made perfectly clear. He

would remark that, with reference to the range of multiplier alluded to, although it was quite true that this was much greater in the case of the tangent form of integrator, in fact was infinitely great, yet for all practical purposes the sine form enabled quite sufficient range to be obtained while the latter had the great advantage of allowing of the polar instead of the linear form of construction. He wished to say in conclusion, that the point about which he felt extremely gratified was that none of the speakers, although many of them had had considerable experience concerning integrators, had taken any objection to the division of the subject he had made. When he began to write the paper, his only object was to endeavor to classify and put in more concise form the details of the subject, but he found the utmost difficulty in doing so. He found out that the other classifications in use would not embrace all the instruments, and were very unsatisfactory; directly, however, the mode of treating the whole question

which he had employed in the paper, occurred to him, the matter began to assume a clearer aspect, and he had found little trouble in grouping and explaining all known integrators by its means; and inasmuch as the classification had led him to certain modifications of his own instruments, he hoped that it would not be altogether without results to other workers on the subject.

Correspondence.

Mr. B. Abdank-Abakanowicz submitted a linear planimeter of precision of his construction (Fig. 58). This instrument was based on the employment of a cylinder C turning freely round its axis. and of a roller *r*, the plane of which could be inclined as wished in respect to the axis of the cylinder, and which could be moved along the generatrix of the cylinder. The displacement of any point of the cylinder was in proportion to the tangent of the angle that the plane of the roller formed with the axis of the cylinder. On a bar DD, fixed on the surface of a drawing, were mounted the pedestals S, S′ carrying between their points the cylinder C. The roller *r* was mounted on a carrier B, movable along the rod XX. Another carriage, A, supported a bar YY, perpendicular to XX. The two carriers were joined together by the rod *g*, and their distance could be varied to change the value of the constant. On the bar DD

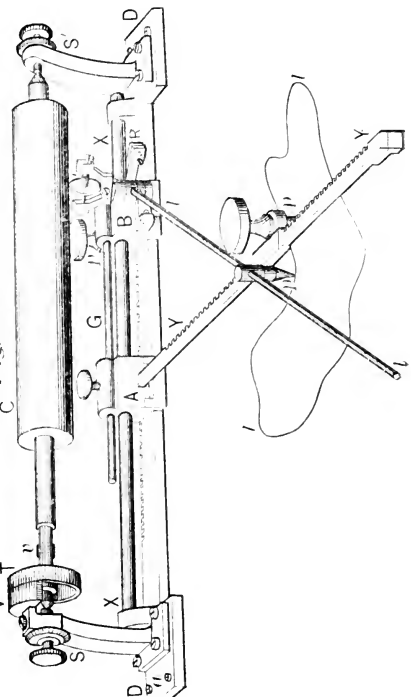


Fig. 52

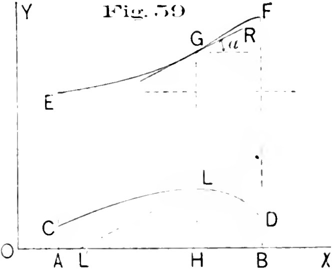
as well as on YY racks were cut, in which worked the pinions p and p'. The rod l, which made the plane of the roller deviate, was placed on the plane of r by means of regulating screws mounted on the arm R. In turning the two pinions p and p', it was easy to follow exactly with the pointer P, the curve bb of which it was desired to find the superficies. There was then imparted to the triangle ABP a movement of translation, and at the same time its height AP was made to vary according to the ordinates of the given curve. The roller was always pressed by a spring against the surface of the cylinder. The number of turns of the cylinder was read on a counter placed on an endless screw v (this counter was not shown on Fig. 58), and the fractions on the drum T supplied with a vernier, V. The integrators of Mr. C. V. Boys were founded on the same principle, and he took the opportunity of stating that Mr. Boys had found this principle of integration independently of him, and that Mr. Boys had recognized the priority of

his invention. In other respects the form of the apparatus of Mr. Boys differed from that he had constructed.

The other apparatus he had made on the same principle, for a special object he was now engaged in pursuing. He sought to make apparatus capable of tracing the integral curve, and it was to this special end he had directed his labors. This was briefly the problem: given the curve CLD (Fig. 59), of which the equation was $y=f(x)$, to trace mechanically another curve EGF, of which the equation should be $Y=\int f(x)\,dx+C$. The constant=AE. Since 1878 he had constructed machines to solve this problem, and he had brought them forward on various occasions. It would occupy too long a time now to explain these instruments. They were made with the object of rendering service, particularly in the art of engineering, and in limits much more extensive than could be accomplished by planimeters. In reality the integral curve could be met with in almost all problems of statics. Those

who were familiar with the elegant processes of graphic statics knew how important to the calculation of bridges, arches, statical moments, and of inertia, was the outline traces by funicular curves. The machines that he had called

Fig. 59



briefly "Integraphs" traced these curves mechanically, and to obtain them it was sufficient to have the same data as for the ordinary methods of graphic statics. It would be possible to arrive at the same result with linear planimeters, but it would be necessary to operate by successive additions, whilst the machines would register at each moment the increment of the integral, tracing a curve on

which could be made all the necessary geometric operations.

Major-General H. P. Babbage remarked that what most interested him was the contrast between arithmetical calculating machines and these integrators. In the first there was absolute accuracy of result, and the same with all operators; and there were mechanical means for correcting, to a certain extent, slackness of the machinery. Friction too had to be avoided. In the other instruments nearly all this was reversed, and it would seem that with the multiplication of reliable calculating machines, all except the simplest planimeters would become obsolete.

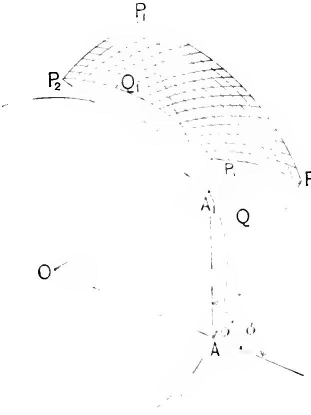
Professor A. G. Greenhill said, suppose O to be the fixed center, OAP the planimeter, A the joint containing the sphere (Fig. 60).

1. Fix the joint A, and move P to P_1 on the arc of a circle, center O; then if OA turned through an angle θ, the dial at A would register an angle M $\theta \cos \varphi$, if the angle between AP and OA pro-

duced was φ, and M was some constant.

2. Fix the joint O, and move P_1 to P_2

Fig. 60



on the arc of a circle, center A_1; the dial at A would not move.

3. Fix the joint A, and move A_1 back to A, and P_2 to P_3; then the dial at A would move back an angle $M\ \theta \cos \varphi'$, if φ' was the angle between AP_3 and OA produced.

4. Fix the joint O, and move P_3 to P on the arc of a circle, center A; the dial would not move.

In completing the circle of the finite area $PP_1P_2P_3$, the dial would then have registered an angle $M\theta$ (cos φ — cos φ').

But the area $PP_1P_2P_3$ = area PP_1Q_1Q

$$=\tfrac{1}{2}\,\theta\,(OP^2-OQ^2)$$

$$=\tfrac{1}{2}\,\theta\,(a^2+2\,ab\cos\varphi+b^2-a^2-2\,ab\cos\varphi'-b^2)$$

$$=a\,b\,\theta\,(\cos\varphi-\cos\varphi'),$$

if $OA=a$. $AP=b$.

Then if $M-ab$, the dial would register the area $PP_1P_2P_3$.

Any irregular area must be supposed to be made up of infinitesimal elements formed in the same manner as $PP_1P_2P_3$.

Mr. E. Sang observed that in the "Transactions of the Royal Scottish Society of Arts" there was a description of a plantometer, by Arthur Beverley, of Dunedin, New Zealand. In it the rubbing wheel was guided in a straight path. A very simple analysis showed that, because the

tracer returns to its first position, the result was true, whatever might be the curve of this guide, so that Beverley's and Amsler-Laffon's were two cases of a general law. They were identical in principle; it was not likely that, in his out-of-the-way position, Beverley had known of the other.

Professor Shaw remarked that he had already called attention to the fact that Mr. Abdank-Abakanowicz had anticipated Mr. Boys in the use of the non-slipping principle for integrators, but he was glad that the former gentleman had alluded to the tracing of an integral curve by instruments acting on this principle. This was a problem of considerable interest, and the author might mention that he had, in a paper before the Royal Society, called attention to a particular case of the integral curve. In this case if the section of any solid were taken on the plane of the paper, a curve, which he had called the curve of areas, might be drawn, the ordinates at any point of which represented the area of the cross-section of the

solid at that point perpendicular to the plane of the paper. Such curves might no doubt be drawn by an adaptation of the instruments of Messrs. Abdank-Abakanowicz and Boys, and there seemed to be considerable scope for further investigation in that direction.

The author was obliged to express his disagreement with the opinion of General Babbage, that all integrators except the simplest planimeters would become obsolete and give place to arithmetical calculating machines. Continuous and discontinuous calculating machines, as they had respectively been called, had entirely different kinds of operations to perform, and there was a wide field for the employment of both. All efforts to employ mere combinations of trains of wheelwork for such operations as were required in continuous integrators had hitherto entirely failed, and the author did not see how it was possible to deal in this way with the continuously varying quantities which came into the problem. No doubt the mechanical difficulties were

great, but that they were not insuperable was proved by the daily use of the disk, globe, and cylinder of Professor James Thomson in connection with tidal calculations and meteorological work, and, indeed, this of itself was sufficient refutation of General Babbage's view.